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ELEMENTS OF GEOMETRY

BY

ANDREW W. PHILLIPS, Ph.D.

AND

IRVING FISHER, Ph.D.

PROFESSORS IN YALE UNIVERSITY

PART ONE—PLANE GEOMETRY

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PREFACE

THE present volume consists of the first five books of the authors' "Elements of Geometry," or that portion which relates to Plane Geometry.

While the book speaks for itself, we would call attention to some of its most important features.

The *Introduction* presents in the shortest possible compass the general outlines of the science to be studied, and leads at once to the actual study itself.

The *definitions* are distributed through the book as they are needed, instead of being grouped in long lists many pages in advance of the propositions to which they apply. An alphabetical index is added for easy reference.

The *constructions* are also distributed, so that the student is taught how to make a figure at the same time that he is required to use it in demonstration.

Extensive use has been made of *natural* and *symmetrical* methods of demonstration. Such methods are used for deducing the formula for the sum of the angles of a triangle, for the sum of the exterior and interior angles of a polygon, for parallel lines, for the theorems on regular polygons, and for similar figures.

The *theory of limits* is treated with rigor, and not passed over as self-evident.

Attention is also called to the theorems of *proportion* and the use of *corollaries* as *exercises* to supply the need of "inventional geometry."

We would here express our grateful acknowledgments to all who have aided in the preparation of this book; to Miss Elizabeth H. Richards, whose successful experience in fitting students for college in Plane Geometry has rendered her criticisms and suggestions most valuable; and to our colleagues, Messrs. W. M. Strong and Joseph Bowden, Jr. Mr. Strong has selected, for the most part, the exercises at the end of the book, and Mr. Bowden has examined critically the references and proof-sheets of the book.

ANDREW W. PHILLIPS,
IRVING FISHER.

YALE UNIVERSITY.

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SPECIAL TERMS

An **axiom** is a truth assumed as self-evident.

A **theorem** is a truth which becomes evident by a train of reasoning called a **demonstration.**

A theorem consists of two parts, the *hypothesis*, that which is given, and the *conclusion*, that which is to be proved.

A **problem** is a question proposed which requires a solution.

A **proposition** is a general term for either a theorem or problem.

One theorem is the **converse** of another when the conclusion of the first is made the hypothesis of the second, and the hypothesis of the first is made the conclusion of the second.

The converse of a truth is not always true. Thus, "If a man is in New York City he is in New York State," is true; but the converse, "If a man is in New York State he is in New York City," is not necessarily true.

When one theorem is easily deduced from another the first is sometimes called a **corollary** of the second.

A theorem used merely to prepare the way for another theorem is sometimes called a **lemma.**

SYMBOLS AND ABBREVIATIONS

$+$ plus.

$-$ minus.

$>$ is greater than.

$<$ is less than.

$\times$ multiplied by.

$=$ equals.

≎ is equivalent to.

Alt.-int.—Alternate interior.

Ax.—Axiom.

Cons.—Construction.

Cor.—Corollary.

Def.—Definition.

Fig.—Figure.

Hyp.—Hypothesis.

Iden.—Identical.

Q. E. D.—Quod erat demonstrandum.

Q. E. F.—Quod erat faciendum.

Sup.-adj.—Supplementary adjacent.

GEOMETRY

INTRODUCTION

FUNDAMENTAL CONCEPTIONS

1. *Def.*—**Geometry** is the science of **space.**

2. Every one has a notion of space extending indefinitely in all directions. Every material body, as a rock, a tree, or a house, occupies a limited portion of space. The portion of space which a body occupies, considered separately from the matter of which it is composed, is a *geometrical solid.* The material body is a *physical solid.* Only geometrical solids are here considered, and they are called simply *solids.*

Def.—A **solid** is, then, a limited portion of **space.**

3. *Def.*—The boundaries of a solid are **surfaces** (that is, the surfaces separate it from the surrounding space).

A surface is no part of a solid.

4. *Def.*—The boundaries of a surface are **lines.**

A line is no part of a surface.

5. *Def.*—The boundaries (or ends) of a line are **points.**

A point is no part of a line.

6. The solid, surface, line, and point are the four fundamental conceptions of geometry. They may also be considered in the reverse order, thus:

(1.) A **point** has position but no magnitude.

(2.) If a point moves, it generates (traces) a **line.**

This motion gives to the line its only magnitude, *length.*

(3.) If a line moves (not along itself), it generates a **surface.**

This motion gives to the surface, besides length, *breadth.*

(4.) If a surface moves (not along itself), it generates a **solid.**

This motion gives to the solid, besides length and breadth, *thickness.*

Def.—A **figure** is any combination of points, lines, surfaces, or solids.

7. *Def.*—A **straight line** is a line which is the shortest path between any two of its points.

8. *Def.*—A **plane surface** (or simply a **plane**) is a surface such that, if any two points in it are taken, the straight line passing through them lies wholly in the surface.

9. *Def.*—Two straight lines are **parallel** which lie in the same plane and never meet, however far produced.

GEOMETRIC AXIOMS

10. All the truths of geometry rest upon three fundamental axioms, viz.:

(*a.*) **Straight line axiom.**—Through every two points in space there is one and only one straight line.

This is sometimes expressed as follows: Two points *determine* a straight line.

(*b.*) **Parallel axiom.**—Through a given point there is one and only one straight line parallel to a given straight line.

(*c.*) **Superposition axiom.**—Any figure in a plane may be freely moved about in that plane without change of size or shape. Likewise, any figure in space may be freely moved about in space without change of size or shape.

GENERAL AXIOMS

11. In reasoning from one geometric truth to another the following general axioms are also employed, viz.:

(1.) Things equal to the same thing are equal to each other.

(2.) If equals be added to equals, the wholes are equal.

(3.) If equals be taken from equals, the remainders are equal.

(4.) If equals be added to unequals, the wholes are unequal in the same order.

(5.) If equals be taken from unequals, the remainders are unequal in the same order.

(6.) If unequals be taken from equals, the remainders are unequal in the opposite order.

(7.) If equals be multiplied by equals, the products are equal; and if unequals be multiplied by equals, the products are unequal in the same order.

(8.) If equals be divided by equals, the quotients are equal; and if unequals be divided by equals, the quotients are unequal in the same order.

(9.) If unequals be added to unequals, the lesser to the lesser and the greater to the greater, the wholes will be unequal in the same order.

(10.) The whole is greater than any of its parts.

(11.) The whole is equal to the sum of all its parts.

(12.) If of two unequal quantities the lesser increases (continuously and indefinitely) while the greater decreases; they must become equal once and but once.

(13.) If of three quantities the first is greater than the second and the second greater than the third, then the first is greater than the third.

12. *Def.*—**Plane Geometry** treats of figures in the same plane.

13. *Def.*—**Solid Geometry,** or the **geometry of space,** treats of figures not wholly in the same plane.

PLANE GEOMETRY

BOOK I

FIGURES FORMED BY STRAIGHT LINES

14. *Defs.*—An **angle** is a figure formed by two straight lines diverging from the same point.

This point is the *vertex* of the angle, and the lines are its *sides*.

A clear notion of an angle may be obtained by observing the hands of a clock, which form a continually varying angle.

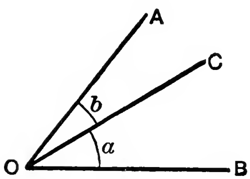


FIG. 1

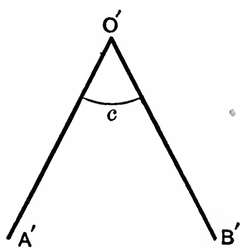


FIG. 2

We may designate an angle by a letter placed within as a and b in Fig. 1, and c in Fig. 2.

Three letters may be used, viz.: one letter on each of its sides, together with one at the vertex, which must be written between the other two, as AOC, BOC, and AOB in Fig. 1, and $A'O'B'$ in Fig. 2.

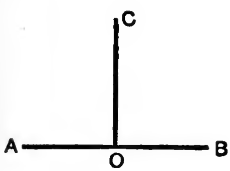
If there is but one angle at a point, it may be denoted by a single letter at that point, as O' in Fig. 2.

Angles with a common vertex and side, as a and b, are said to be **adjacent**.

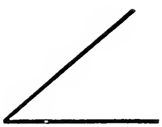
15. *Def.*—Two angles are **equal** if they can be made to coincide. Also, in general, any two figures are equal which can be made to coincide.

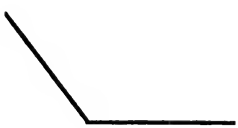
Thus, suppose we place the angle AOB on the angle $A'O'B'$ so that O shall fall at O', and the side OA along $O'A'$; then, if the side OB also falls along $O'B'$, the angles are equal, *whatever may be the length of each of their sides.*

16. *Def.*—When one straight line is drawn from a point in another, so that the two adjacent angles are equal, each of these angles is a **right angle**, and the lines are **perpendicular.**



RIGHT ANGLES


ACUTE ANGLE


OBTUSE ANGLE

Thus, if the angles AOC and COB are equal, they are right angles, and CO is perpendicular to AB.

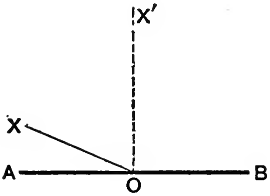
When a straight line is perpendicular to another straight line, its point of intersection with the second line is called the **foot of the perpendicular.**

17. *Def.*—An **acute** angle is an angle less than a right angle; an **obtuse** angle, greater.

The term **oblique** angle may be applied to any angle which is not a right angle.

PROPOSITION I. THEOREM

18. *From a point in a straight line one perpendicular, and only one, can be drawn (on the same side of the given straight line).*



GIVEN a straight line, *AB*, and any point, *O*, upon it.

TO PROVE—from *O* one, and only one, perpendicular can be drawn to *AB* (on the same side of *AB*).

Suppose a straight line *OX* to revolve about *O*. Ax. *c*

In every one of its successive positions it forms two different angles with the line *AB*, viz.: *XOA* and *XOB*.

As it revolves from the position *OA* around to the position *OB* the lesser angle, *XOA*, will continuously increase, and the other, *XOB*, will continuously decrease.

There must, therefore, be one and only one position of *OX*, as *OX′* where the angles become equal. Ax. 12

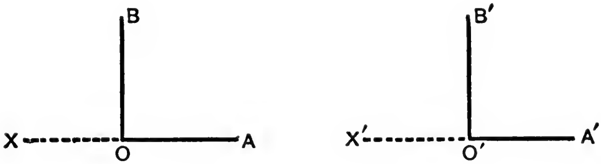
[If, of two unequal quantities, the lesser increases, etc.]

That is, there must be one and only one perpendicular to *AB* at *O*. Q. E. D.

Question.—The above proposition applies to the plane of the diagram. Could you draw any other lines perpendicular to *AB* at *O* *out* of the plane of the page?

PROPOSITION II. THEOREM

19. *All right angles are equal.*



GIVEN any two right angles AOB and $A'O'B'$.

TO PROVE they are equal.

Apply $A'O'B'$ to AOB so that the vertex O' shall fall on O, and so that A', any point in one side of $A'O'B'$, shall fall on some point in OA or OA produced.

Then the line $O'A'$ will coincide with OA, even if both be produced indefinitely. Ax. *a*

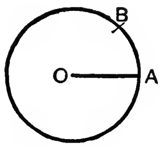
[Two points determine a straight line.]

If $O'B'$ should not fall along OB, there would be two lines, $O'B'$ and OB, perpendicular to the same line from the same point, which is impossible. § 18

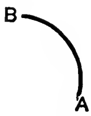
[From a point in a straight line, one perpendicular, and only one can be drawn.]

Therefore $O'B'$ must fall along OB—that is, the angles $A'O'B'$ and AOB coincide and are equal. Q. E. D.

20. *Defs.*—A **circle** is a figure bounded by a line all points of which are equally distant from a point within called the **centre.**



CIRCLE



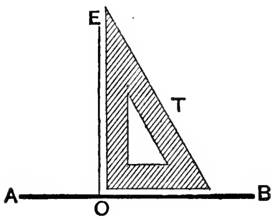
ARC

The bounding line is called the **circumference.**

Any portion of the circumference is called an **arc.**

Any one of the equal lines from the centre to the circumference (as OA) is called a **radius.**

21. CONSTRUCTION. *To draw a perpendicular from a straight line AB at some point in it, as O.*

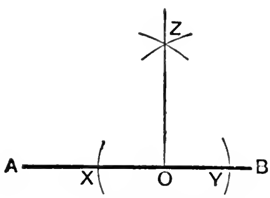


FIRST METHOD

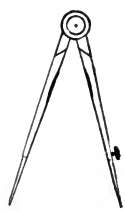
First method.—Place a right-angled ruler T with the vertex of its right angle at O and one of its edges along AB. Draw OE along its other edge. OE will be the required line for, first, it is drawn through O, and, second, it is drawn perpendicular to AB.

The student should observe that it is impossible to construct an absolutely accurate diagram, for no ruler is absolutely accurate nor can it be applied with absolute accuracy. Moreover the dots and marks formed by a pencil, however well sharpened, are not absolute points and lines, for the dots have *some* magnitude, and the marks *some* breadth. Diagrams only *approximate* the ideal points and lines intended.

If, however, the practical means employed *could be made perfect*, the resulting construction *would be* absolutely exact. Hence we may say of the preceding construction, the *method* is perfect, though the *means* can never be. This method is largely used by draughtsmen and carpenters.



SECOND METHOD



COMPASSES

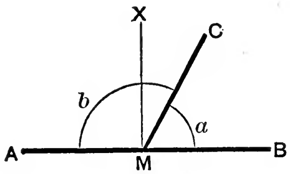
Second method (with straight ruler and compasses).—Take O as a centre, and with any convenient radius describe with the compasses two arcs cutting AB at X and Y. Then with X and Y as centres, with a somewhat longer radius describe two arcs cutting each other at Z. Join OZ with the ruler. OZ will be the perpendicular required.

[The correctness of the second method can be proved after reaching § 89.]

Of the two methods above described, the first has the advantage of quickness, but it assumes that the ruler is really made with a right angle, that is, it assumes that some one has already constructed a right angle and all we do is to copy it. The second method is free from this assumption, though, in both methods, it is assumed that the ruler is made with a straight edge, that is, that some one has already constructed a straight line. The first way of constructing a straight line was by stretching a string, a method still used by carpenters. In fact the word "straight" originally meant "stretched." The ancient Egyptians used this method, and even invented a way of making a right angle by stretching a cord. (See foot-note to § 317.)

PROPOSITION III. THEOREM

22. *The two angles which one straight line makes with another, upon one side of it, are together equal to two right angles.*



GIVEN—the straight line CM meeting the straight line AB at M and forming the angles a and b.

TO PROVE $a+b=2$ right angles.

Suppose MX drawn perpendicular to AB. § 18

[From a point in a straight line one perpendicular can be drawn.]

Then $BMX+XMA=2$ right angles. § 16

We may substitute for BMX its equal, $a+CMX$. Ax. 11

[A whole is equal to the sum of its parts.]

This gives $a+CMX+XMA=2$ right angles.

We may now substitute for $CMX+XMA$ the angle b.

[Same axiom.]

This gives $a+b=2$ right angles. Q. E. D.

23. *Defs.*—Two angles whose sum is equal to a right angle, are **complementary** angles.

Two angles whose sum is two right angles, are **supplementary** angles.

The two angles which one straight line makes with another on one side of it (as a and b), are **supplementary-adjacent** angles.

24. COR. I. *If one of the angles formed by the intersection of two straight lines is a right angle, the others are right angles.* (Fig. 1.)

Hint.—Apply Proposition III.

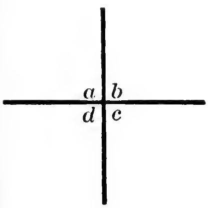


FIG. 1

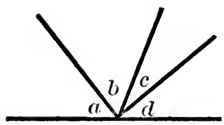


FIG. 2

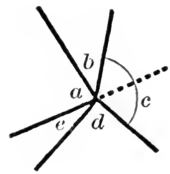


FIG. 3

25. COR. II. *If of two intersecting straight lines one is perpendicular to the other, then the second is also perpendicular to the first.*

Hint.—Apply Corollary I.

26. In COROLLARIES the proof is left, wholly or in part, to the student. Practice will give him the power of carefully stating and separating the steps and *finding for each a satisfactory reason.*

27. COR. III. *The sum of all the angles about a point on one side of a straight line equals two right angles.* (Fig. 2.)

Hint.—Group the angles into two angles and apply Proposition III.

28. COR. IV. *The sum of all the angles about a point equals four right angles.* (Fig. 3.)

Hint.—Prolong one of the lines through the vertex, separating the opposite angle *c* into two angles, and apply Corollary III.

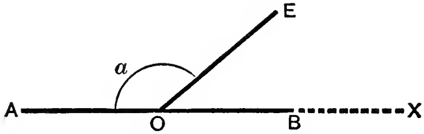
Question.—If, of three angles around a point, two are each one and a third right angles, how much is the third angle?

Question.—If six angles about a point are all equal, how large is each angle?

PROPOSITION IV. THEOREM

29. *If two adjacent angles are together equal to two right angles, their exterior sides are in the same straight line.*

[The converse of Proposition III.]



GIVEN $a + EOB = 2$ right angles.

TO PROVE AO and OB form one straight line.

Let OX be the prolongation of AO.

$a + EOB = 2$ right angles. Hyp.

$a + EOX = 2$ right angles. § 22

[Being sup.-adj.]

Hence $a + EOB = a + EOX$. Ax. 1

Subtracting a, $EOB = EOX$. Ax. 3

Hence OB must coincide with OX.

Otherwise one of the angles (EOB and EOX) would include the other, and they could *not* be equal. Ax. 10

Therefore OB lies in the same straight line with OA.

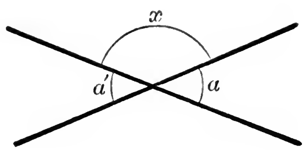
Q. E. D.

Question.—If two angles are supplementary-adjacent, and their difference is one right angle, how large is each?

Question.—The angles on the same side of a straight line are three in number. The greatest is three times the least, and the remaining one is twice the least. How large is each? In how many ways can they be arranged on the straight line?

PROPOSITION V. THEOREM

30. *If two straight lines intersect, the opposite (or vertical) angles are equal.*



GIVEN—two intersecting straight lines forming the opposite angles a and a'.

TO PROVE $a = a'$.

$a + x = 2$ right angles. § 22

$a' + x = 2$ right angles. § 22

[Being, in each case, sup.-adj.]

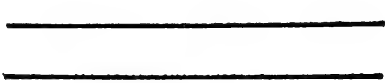
Therefore $a + x = a' + x$. Ax. 1

Subtracting x, $a = a'$. Ax. 3

Q. E. D.

PARALLEL LINES AND SYMMETRICAL FIGURES

31. *Def.*—Two straight lines are **parallel** which lie in the same plane, but never meet, however far produced.



PARALLEL LINES

32. *Def.*—Two figures are **symmetrical with respect to a straight line** called an **axis of symmetry,** when, if one of them be folded over on that line as an axis, it will coincide with the other. (Fig. 1.)

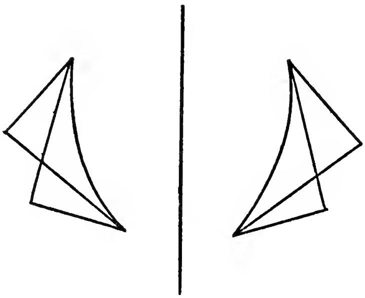


FIG. 1

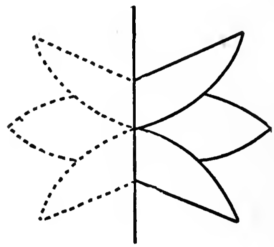
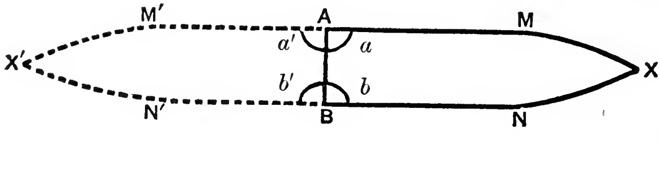


FIG. 2

A clear notion of this kind of symmetry may be obtained by drawing any figure in ink, and before the ink has dried folding the paper on to itself over a crease. The original figure and the resulting impression are symmetrical with respect to the crease as an axis. (Fig. 2.)

PROPOSITION VI. THEOREM

33. *Two straight lines perpendicular to the same straight line are parallel.*



GIVEN AM and BN perpendicular to AB.

TO PROVE AM and BN parallel.

If AM and BN should meet, either at the right or left, as at X, fold the figure AXB about AB as an axis to form the symmetrical impression $AX'B$, the right angles a and b forming the impressions a' and b' respectively.

Then AM and AM' form one and the same straight line,
and BN and BN' form one and the same straight line. § 29

[If two adjacent angles (as a' and a) are together equal to two right angles, their exterior sides are in the same straight line.]

Hence we would have two straight lines through X and X', which is absurd. Ax. *a*

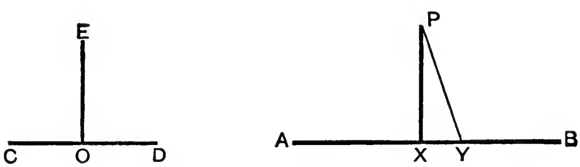
[Two points determine a straight line.]

Therefore AM and BN cannot meet, and, as they lie in the same plane, they must be parallel. § 31

Q. E. D.

Question.—Will the preceding proposition still be true if the lines are not all confined to one plane?

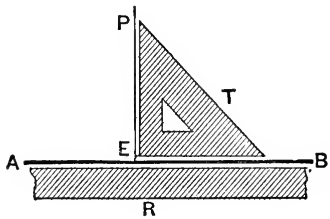
34. COR. *Through a given point P without the line one and only one perpendicular can be drawn to a given straight line, AB.*



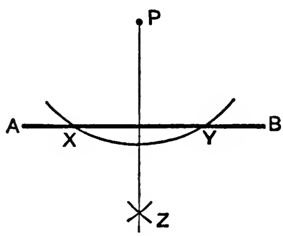
OUTLINE PROOF: From O in another line CD erect a perpendicular OE. (By what authority?) Superpose CD upon AB, and move it along AB until OE contains P. (What axiom applies?)

Second, suppose two were possible, as PX and PY, and show that this would contradict Proposition VI.

35. CONSTRUCTION. *To drop a perpendicular to a straight line AB from a point P without the line.*



First method.—Apply a straight edge of a ruler R to the straight line AB. Place one side of a right-angled ruler T upon the ruler R, making another side perpendicular to AB. Then slide T along AB until the perpendicular edge contains P. Draw PE along that edge. PE is the perpendicular required, for it is drawn through P and is perpendicular to AB.



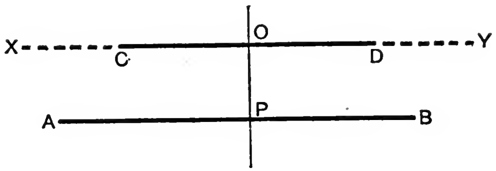
Second method.—From P as a centre with a convenient radius describe an arc cutting AB at X and Y. Then with X and Y in turn as centres describe arcs with equal radii intersecting at Z. Join PZ. This will be the required perpendicular.

[This can be proved correct after reaching § 104.]

PROPOSITION VII. THEOREM

36. *If two straight lines are parallel, and a third straight line is perpendicular to one of them, it is perpendicular to the other.*

[Converse of Proposition VI.]



GIVEN—*CD* and *AB* parallel, and *PO* perpendicular to *AB*.

TO PROVE *PO* perpendicular to *CD*.

Suppose *XY* to be drawn through *O* perpendicular to *OP*.

Then *XY* is parallel to *AB*. § 33

[Two straight lines perpendicular to the same straight line are parallel.]

But *CD* is parallel to *AB*. Hyp.

Hence *CD* must coincide with *XY*. Ax. *b*

[Through any point there is one and only one straight line parallel to a given straight line.]

That is *CD* must be perpendicular to *PO*,

and *OP* is perpendicular to *CD*. § 25

Q. E. D.

37. CONSTRUCTION. *To draw a straight line through a given point C parallel to a given straight line AB.*

First method (Fig. 1).—Place a right-angled ruler in the position *T*, making one edge about the right angle coincident with *AB*, and along the other edge place a ruler *R*.

Then hold the ruler *R* firmly against the paper. Slide *T* to the position *T'* till its edge reaches *C*. Draw *CX*. It is the parallel required. (Why?)

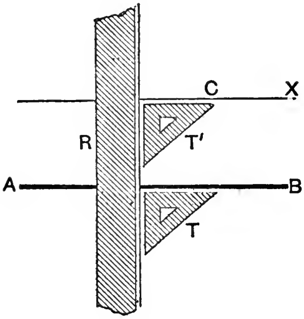


FIG. 1

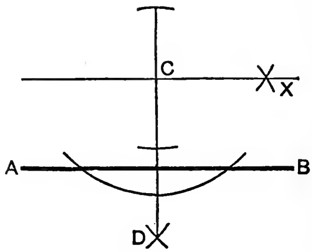


FIG. 2

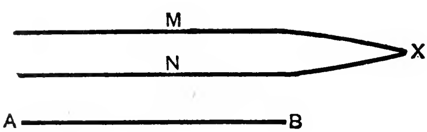
Second method (Fig. 2).—From *C* draw *CD* perpendicular to *AB*. § 35

At *C* draw *CX* perpendicular to *CD*. § 21

Then *CX* is the required parallel to *AB*. (Why?)

PROPOSITION VIII. THEOREM

38. *If two straight lines are parallel to a third straight line, they are parallel to each other.*



GIVEN *M* and *N* each parallel to *AB*.

TO PROVE *M* and *N* parallel to each other.

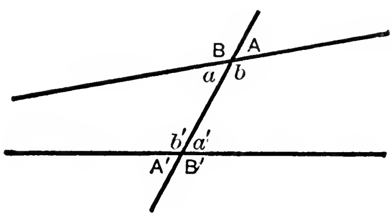
If M and N should meet, as at X, we would have two parallels to AB through the same point X, which is absurd. Ax. *b*

[Through one point there is one and only one straight line parallel to a given straight line.]

Therefore M and N cannot meet, and, lying in the same plane, must be parallel. § 31

Q. E. D.

39. *Defs.*—When two straight lines are cut by a third straight line, of the eight angles formed—



a, b, a', b', are **interior** angles.

A, B, A', B', are **exterior** angles.

a and a', or b and b', are **alternate-interior** angles.

A and A', or B and B', are **alternate-exterior** angles.

A and a', b and B', B and b', or a and A', are **corresponding** angles.

Question.—Of the eight angles, which are always equal, and why?

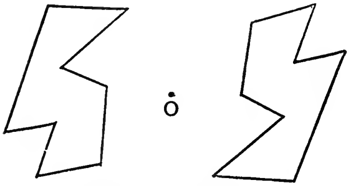
Question.—If $A = A'$, what other angles are also equal to A, and why? Are the remaining angles all equal, and if so, why?

Question.—If $A = A'$ and also $A = B$, what angles are equal, and why?

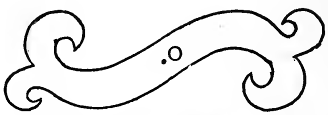
40. *Defs.*—Two figures are **symmetrical with respect to a point** called the **centre of symmetry** when, if one of them is revolved half way round on this point as a pivot, it will coincide with the other.

A single figure is said to be symmetrical with respect to a point called the centre of symmetry if, when the figure is turned half way round on this point as a pivot, each portion of the figure will take the position previously occupied by another part.

[A figure is said to be turned half way round a point when a line through the point turns through two right angles.]



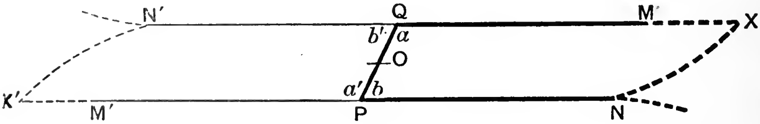
TWO FIGURES SYMMETRICAL WITH RESPECT TO O



A SINGLE FIGURE SYMMETRICAL WITH RESPECT TO O

PROPOSITION IX. THEOREM

41. *When two straight lines are cut by a third straight line, if the two interior angles on the same side of the cutting line are together equal to two right angles, then the two straight lines are parallel.*



GIVEN—PQ cutting QM and PN so that a and b on the same side of PQ are together equal to two right angles.

TO PROVE QM and PN parallel.

About O, the middle point of PQ, as a pivot, revolve the figure $QMXNP$ half way round to the symmetrical position $PM'X'N'Q$, so that P and Q exchange places.

The angle a is the supplement of b. Hyp.

Hence, when a takes the position a', PM' must be the prolongation of PN. § 29

[If two adjacent angles equal two right angles, their exterior sides form the same straight line.]

Likewise QN' is the prolongation of QM.

Now if these lines should meet on the right of PQ, as at X, they would also meet on the left, at X'. § 40

And we would have two straight lines between the two points, X and X', which is absurd. Ax. *a*

If they do *not* meet on the right of PQ, neither can they meet on the left of it. § 40

Hence QM and PN do not meet, and, being in the same plane, are parallel. Q. E. D.

It may be observed that the preceding proposition rests on only *two* of the three geometric axioms stated in § 10, viz.: the *superposition axiom*, assumed in turning the figure unchanged about O, and the *straight-line axiom*, used to prove that there cannot be two straight lines between X and X'. The *parallel axiom* (viz.: that through a point only one straight line can be drawn parallel to a given straight line) has only been used so far in Propositions VII. and VIII. Mathematicians have tried to dispense with the parallel axiom entirely, but have not succeeded. In fact, Lobatchewsky in 1829 proved that we can never get rid of the parallel axiom without assuming the space in which we live to be very different from what we know it to be through experience. Lobatchewsky tried to imagine a different sort of universe in which the parallel axiom would not be true. This imaginary kind of space is called *non-Euclidean* space, whereas the space in which we really live is called *Euclidean*, because Euclid (about 300 B.C.) first wrote a systematic geometry of our space. In Lobatchewsky's space, Proposition IX. would be true, but Propositions VII. and VIII. would not be true, nor would §§ 47, 48, 49, 51, 58, etc., in Book I., and §§ 284, 327, 329, etc., in Book III.

42. CONSTRUCTION. *To bisect a given straight line, AB.*

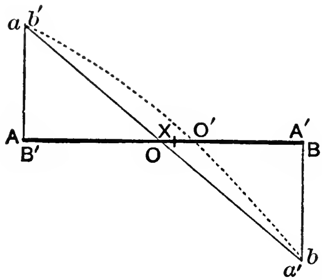


FIG. 1

First method (Fig. 1).—At A and B erect Aa and Bb equal perpendiculars on opposite sides of AB. Join ab cutting AB at O. O is the required middle point.

Proof.—Suppose the middle point of AB is not O, but some other point as X.

Then turn the whole figure about X until AX coincides with its equal BX, A falling on B (call this position of A, A'), and B on A (call this position of B, B'). And O will assume the position O' on the opposite side of X.

Then the perpendicular Aa will fall along Bb. § 18

[From a point in a straight line only one perpendicular can be drawn.]

And a will fall on b (call this position of a, a').

[Since Aa is equal to Bb.]

Likewise b will fall on a (call this position of b, b').

Then the straight line aOb takes the position $a'O'b'$.

That is, through two points, a and b, there would be two straight lines, which is absurd. Ax. a

Hence the supposition that O is not the middle point is false, and O is the middle point. Q. E. D.

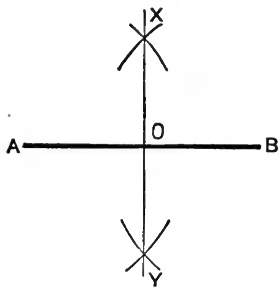


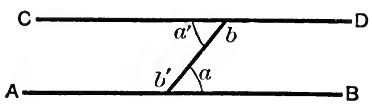
FIG. 2

Second method (Fig. 2).—From A and B as centres with the same radius describe arcs intersecting at X and Y. Join XY intersecting AB at O, the required middle point.

[This method can be proved correct after reaching § 104.]

PROPOSITION X. THEOREM

43. *If two straight lines are cut by a third straight line, making the alternate-interior angles equal, the lines are parallel.*



GIVEN $a = a'$.

TO PROVE AB and CD parallel.

$a' + b = 2$ right angles. § 22

[Being sup.-adj.]

Substitute for a' its equal a.

Then $a + b = 2$ right angles.

Therefore AB is parallel to CD. § 41

[When two straight lines are cut by a third straight line, if the two interior angles on the same side of the cutting line are together equal to two right angles, then the two straight lines are parallel.] Q. E. D.

44. COR. I. *If two or more straight lines are cut by a third, so that corresponding angles are equal, the straight lines are parallel.*

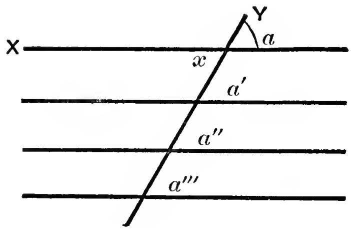


FIG. 1

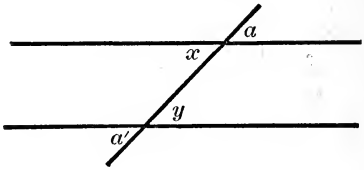
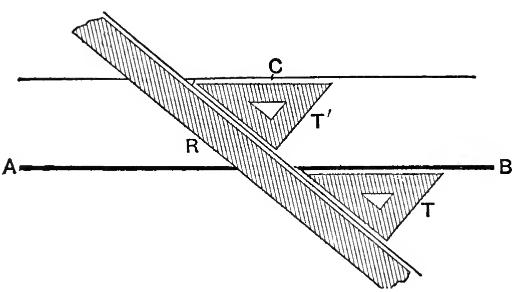


FIG. 2

Hint.—Reduce to Proposition X. by means of Proposition V.

45. COR. II. *If two straight lines are cut by a third straight line so that the alternate-exterior angles are equal, the lines are parallel.*

Hint.—Reduce to Proposition X. by Proposition V.

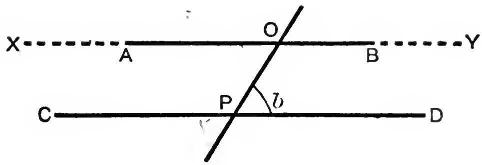


46. *Exercise.*—Show by § 44 that the construction of § 37 may be effected as in the preceding figure.

PROPOSITION XI. THEOREM

47. *If two parallel lines are cut by a third straight line, the sum of the two interior angles on the same side of the cutting line is two right angles.*

[Converse of Proposition IX.]



GIVEN—AB and CD parallel and cut by the straight line OP.

TO PROVE $b + POB = 2$ right angles.

Suppose XY to be a line drawn through O, making

$b + POY = 2$ right angles.

Then XY is parallel to CD. § 41

[When two straight lines are cut by a third straight line, if the two interior angles on the same side of the cutting line are together equal to two right angles, the two straight lines are parallel.]

But AB is parallel to CD. Hyp.

Hence AB coincides with XY. Ax. *b*

[Through a given point only one straight line can be drawn parallel to a given straight line.]

And $POB = POY$. Coinciding

Hence $b + POB = b + POY$. Ax. 2

But $b + POY = 2$ right angles. Cons.

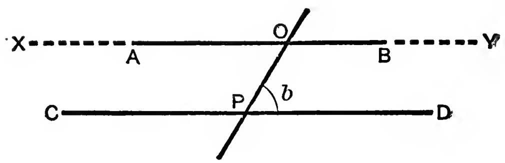
Hence $b + POB = 2$ right angles. Ax. 1

Q. E. D.

PROPOSITION XII. THEOREM

48. *If two parallel lines are cut by a third straight line, then the alternate-interior angles are equal.*

[Converse of Proposition X.]



GIVEN AB and CD parallel.

TO PROVE $b = AOP$.

Suppose XY to be a line drawn through O, making $XOP = b$.

Then XY is parallel to CD. § 43

[If two straight lines are cut by a third straight line, making the alternate-interior angles equal, the lines are parallel.]

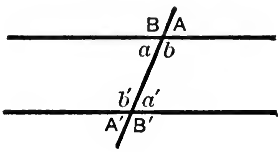
But	AB is parallel to CD.	Hyp.
Hence	AB coincides with XY.	Ax. b
And	$AOP = XOP$.	Coinciding
But	$b = XOP$.	Hyp.
Therefore	$AOP = b$.	Ax. 1

Q. E. D.

49. COR. *If two or more parallel lines are cut by a third straight line, the corresponding angles are equal.*

Hint.—Reduce to Proposition XII.

50. *Remark.*—It follows from the previous propositions and corollaries that if two lines are parallel and cut by a third straight line, as in the figure,



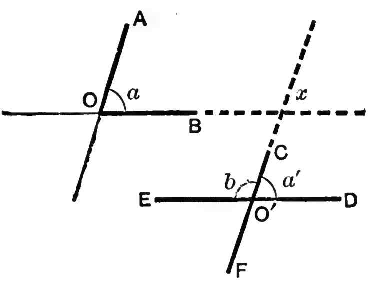
then $A=a=a'=A'$,

$B=b=b'=B'$,

and any angle of the first set is supplementary to any angle of the second set.

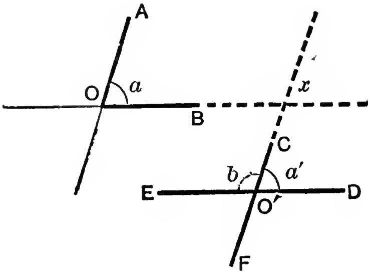
PROPOSITION XIII. THEOREM

51. *Two angles whose sides are parallel, each to each, are either equal or supplementary.*



GIVEN—the angles at O and O' with their sides OA and OB respectively parallel to CF and ED.

TO PROVE the angle $a=a'$, and $a+b=2$ right angles.



Produce OB and $O'C$ until they intersect.

Then $\left.\begin{array}{l} a=x \\ a'=x \end{array}\right\}$ § 49

[Being corresponding angles of parallel lines.]

Therefore $a=a'$. Ax. 1

Moreover, $a'+b=2$ right angles. § 22

Substituting a for its equal a',

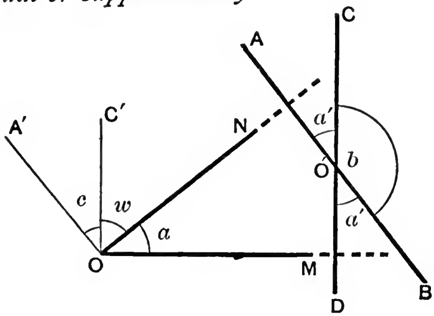
$a+b=2$ right angles. Q. E. D.

52. *Remark.*—To determine when the angles are equal and when supplementary, we observe that every angle, viewed from its vertex, has a *right* and a *left* side. (Thus OA is the left side of a.) Now, if the two angles have the right side of one parallel to the right side of the other and likewise their left sides parallel, they are equal; whereas, if the right side of each is parallel to the left side of the other, they are supplementary. Or, briefly, if their parallel sides are in the *same* right-and-left *order*, they are equal, if in *opposite order*, supplementary.

Thus, a and $EO'F$, which have their sides parallel, right to right (OB to $O'E$) and left to left (OA to $O'F$), are equal, while a and $EO'C$, which have their sides parallel right to left (OB to $O'E$) and left to right (OA to $O'C$), are supplementary. The student can easily test and verify all the sixteen cases obtained by comparing each of the four angles about O with each of the four about O'.

PROPOSITION XIV. THEOREM

53. *Two angles whose sides are perpendicular, each to each, are either equal or supplementary.*



GIVEN—the angle NOM, or a, and the lines AB and CD intersecting at O' and respectively perpendicular to ON and OM.

TO PROVE—the angle $a = a'$, and $a + b = 2$ right angles.

At O, draw OA' parallel to AB and OC' parallel to CD.

OA', being parallel to AB, is perpendicular to ON. § 36

[If two straight lines are parallel, and a third straight line is perpendicular to one of them, it is perpendicular to the other.]

For the same reason OC', being parallel to CD, is perpendicular to OM.

From each of the right angles $A'ON$ and $C'OM$ take away the common angle w.

This leaves $c = a$. Ax. 3

But $c = a'$. § 51

[Having their sides respectively parallel, and in the same right-and-left order.]

Therefore $a = a'$. Ax. 1

Moreover $a' + b = 2$ right angles. § 22

[Being supplementary-adjacent.]

Substituting a for its equal a',

$a + b = 2$ right angles. Q. E. D.

54. *Remark.*—The angles are equal if their sides are perpendicular right to right and left to left, but supplementary if their sides are perpendicular in opposite right-and-left order.

Thus a and $DO'B$, which have their right sides (OM and $O'D$) perpendicular and their left sides (ON and $O'B$) perpendicular, are equal; etc., etc.

TRIANGLES

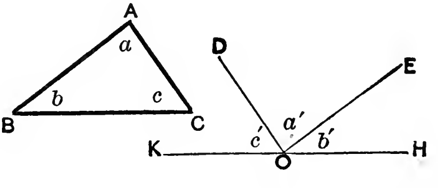
55. *Def.*—A **triangle** is a figure bounded by three straight lines called its **sides.**

56. *Def.*—A **right** triangle is a triangle one of whose angles is a right angle.

57. *Def.*—An **equiangular** triangle is one whose angles are all equal.

PROPOSITION XV. THEOREM

58. *The sum of the three angles of any triangle is two right angles.**



GIVEN ABC, any triangle, with a, b, and c its angles.

TO PROVE $a+b+c=2$ right angles.

Draw KH parallel to BC, and from O, any point of this line, draw OE and OD parallel respectively to the sides AB and AC.

* This was first proved by Pythagoras or his followers about 550 B.C.

Then $\left.\begin{array}{l} a=a' \\ b=b' \\ c=c' \end{array}\right\}$ § 51

[Having their sides parallel and in the same right-and-left order.]

Hence $a+b+c=a'+b'+c'$. Ax. 2

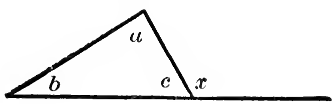
But $a'+b'+c'=2$ right angles. § 27

[The sum of all the angles about a point on one side of a straight line equals two right angles.]

Hence $a+b+c=2$ right angles. Ax. 1

Q. E. D.

59. COR. I. *If one side of a triangle be produced, the exterior angle thus formed equals the sum of the two opposite interior angles (and hence is greater than either of them).*



OUTLINE PROOF: $a+b+c=2$ right angles $=x+c$, whence $a+b=x$. [Give reasons.]

60. COR. II. *If the sum of two angles of a triangle be given, the third angle may be found by taking the sum from two right angles.* [What axiom applies?]

61. COR. III. *If two angles of one triangle are equal respectively to two angles of another triangle, the third angles will be equal.* [What two axioms apply?]

62. COR. IV. *A triangle can have but one right angle, or one obtuse angle.*

63. COR. V. *In a right triangle the sum of the two angles besides the right angle is equal to one right angle.*

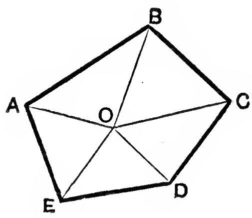
64. COR. VI. *In an equiangular triangle, each angle is one-third of two right angles, and hence two-thirds of one right angle.*

65. *Defs.*—A **polygon** is a figure bounded by straight lines called its **sides.**

A polygon is **convex,** if no straight line can meet its sides in more than two points.

PROPOSITION XVI. THEOREM

66. *The sum of all the angles of any polygon is twice as many right angles as the figure has sides, less four right angles.*



GIVEN $ABCDE$, any polygon, having n sides.

TO PROVE—the sum of its angles is $2n-4$ right angles.

From any point O within the polygon draw lines to all the vertices forming n triangles.

The sum of the angles of each triangle is equal to 2 right angles. § 58

Hence the sum of the angles of the n triangles is equal to $2n$ right angles.

But the angles of the polygon make up all the angles of all the triangles except the angles about O, which make 4 right angles. § 28

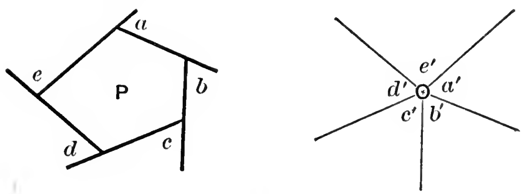
Hence the sum of the angles of the polygon is $2n-4$ right angles. Q. E. D.

67. *Defs.*—A **quadrilateral** is a polygon of four sides, a **pentagon,** of five, a **hexagon,** of six, an **octagon,** of eight, a **decagon,** of ten, a **dodecagon,** of twelve, a **pentedecagon**, of fifteen.

68. *Exercise.*—The sum of the angles of a quadrilateral equals what? of a pentagon? of a hexagon?

PROPOSITION XVII. THEOREM

69. *If the sides of any polygon be successively produced, forming one exterior angle at each vertex, the sum of these exterior angles is four right angles.*



GIVEN—the polygon P with successive exterior angles a, b, c, d, e.

TO PROVE $a+b+c+d+e=4$ right angles.

Through any point O draw lines successively parallel to the sides produced.

Then
$$\left.\begin{aligned} a&=a' \\ b&=b' \\ c&=c' \\ &\text{etc.}\end{aligned}\right\} \qquad \S\ 51$$

[Two angles are equal if their sides are parallel and in the same order.]

Hence $a+b+c+\text{etc.}=a'+b'+c'+\text{etc.}$ Ax. 2

But $a'+b'+c'+\text{etc.}=4$ right angles. § 28

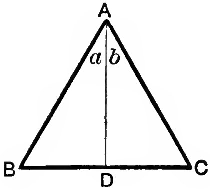
Therefore $a+b+c+\text{etc.}=4$ right angles. Ax. 1

Q. E. D.

70. *Defs.*—An **isosceles** triangle is a triangle two of whose sides are equal. The third side is called the **base.** The opposite vertex is called the **vertex** of the isosceles triangle, and the angle at that vertex the **vertex angle.** An **equilateral** triangle is one whose **three** sides are equal.

PROPOSITION XVIII. THEOREM

71. *The angles at the base of an isosceles triangle are equal.*



GIVEN—the isosceles triangle ABC, AB and AC being equal sides.

TO PROVE the angle B equals the angle C.

Suppose AD to be a line bisecting the angle A.

On AD as an axis revolve the figure ADC till it falls upon the plane of ADB.

AC will fall along AB.

[Since angle $a = b$, by construction.]

C will fall on B.

[Since $AB = AC$, by hypothesis.]

DB will coincide with DC. Ax. *a*

[Their extremities being the same points.]

Hence angle B = angle C. § 15

[Since they coincide.] Q. E. D.

72. COR. I. *The line which bisects the vertex angle of an isosceles triangle bisects the base.*

Hint.—Show where this was proved in the preceding demonstration.

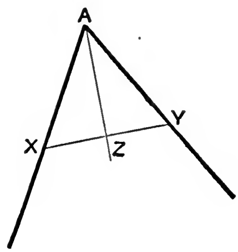
73. COR. II. *The line joining the middle point of the base with the vertex of an isosceles triangle bisects the vertex angle.*

Hint.—If not, draw the line which *does* bisect the vertex angle and prove it coincides with the given line.

74. COR. III. *Every equilateral triangle is also equiangular, and each angle is one-third of two right angles.*

Question.—In how many different ways is an equilateral triangle isosceles?

75. CONSTRUCTION. *To bisect any given angle A.*

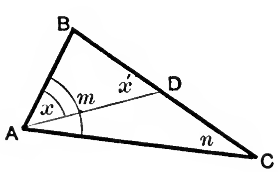


On the sides of the angle, lay off $AX = AY$. Join XY. Bisect XY at Z (§ 42). Join AZ. AZ will bisect the angle A. The student may prove this method correct.

Hint.—Apply one of the preceding corollaries.

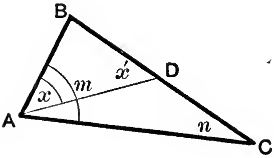
PROPOSITION XIX. THEOREM

76. *If two sides of a triangle are unequal, the opposite angles are unequal, and the greater angle is opposite the greater side.*



GIVEN in the triangle ABC the side $BC >$ side AB.

TO PROVE the angle $m >$ angle n.



On BC take $BD=BA$, and join AD.

Then $x=x'$. § 71

[Being base angles of an isosceles triangle.]

But $x'>n$. § 59

[An exterior angle of a triangle (ADC) is greater than either of the opposite interior angles.]

Substituting x for x', $x>n$.

But $m>x$. Ax. 10

Hence $m>n$. Ax. 13

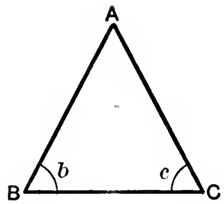
Q. E. D.

OUTLINE PROOF: $m>x=x'>n$, hence $m>n$.

PROPOSITION XX. THEOREM

77. *If two angles of a triangle are equal, the sides opposite are equal—that is, the triangle is isosceles.*

[Converse of Proposition XVIII.]



GIVEN in the triangle ABC, the angle $b=c$.

TO PROVE side $AC=$side AB.

If AC and AB were unequal, b and c would be unequal. § 76

[If two sides of a triangle are unequal the opposite angles are unequal, etc.]

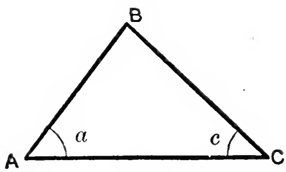
But this contradicts the hypothesis that angle b = angle c.

Hence $AC = AB$. Q. E. D

PROPOSITION XXI. THEOREM

78. *If two angles of a triangle are unequal, the opposite sides are unequal, and the greater side is opposite the greater angle.*

[Converse of Proposition XIX.]



GIVEN in the triangle ABC, the angle $a >$ angle c.

TO PROVE side $BC >$ side AB.

Either AB is equal to, greater than, or less than BC.

If $AB = BC$, then would $c = a$. § 71

[The angles at the base of an isosceles triangle are equal.]

If $AB > BC$, then would $c > a$. § 76

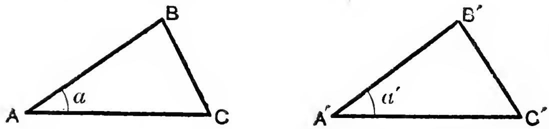
[If two sides of a triangle are unequal, the opposite angles are unequal, and the greater angle is opposite the greater side.]

But both of these conclusions contradict the hypothesis that angle $a >$ angle c.

Therefore $AB < BC$. Q. E. D.

PROPOSITION XXII. THEOREM

79. *If two triangles have two sides and the included angle of one, equal respectively to two sides and the included angle of the other, the triangles are equal.*



GIVEN—*AB*, *AC*, and *a*, of the triangle *ABC* respectively equal to *A'B'*, *A'C'*, and *a'*, of the triangle *A'B'C'*.

TO PROVE the two triangles are equal.

Place *ABC* on *A'B'C'*, making *AB* coincide with its equal *A'B'*

Then, since $a=a'$, the side *AC* will fall along *A'C'*.

Also, since $AC=A'C'$, the point *C* will fall on *C'*.

Then *BC* will coincide with *B'C'*. Ax. *a*

[Having their extremities in the same points.]

And, since the triangles completely coincide, they are equal. § 15

Q. E. D.

80. CONSTRUCTION. *To construct an angle at a given point A' as its vertex, and on a given line A'B' as a side, equal to a given angle BAC at a different vertex A.*

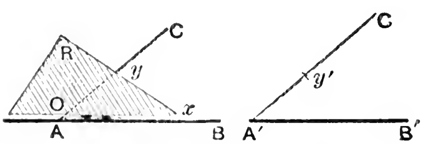


FIG. 1

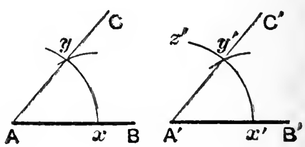


FIG. 2

First method (Fig. 1).—Place a triangular ruler, *R*, so that the straight edge falls along *AB*. Mark *y* on another edge where this edge cuts *AC*. Also mark the point *A* on the ruler and call it *O*. Draw *Oy* on the ruler. Then the angle *BAC* is reproduced on the ruler as *xOy*. Then, placing the ruler with *O* at *A'* and *Ox* along *A'B'*, retransfer the angle *xOy* of the ruler to the paper making *B'A'C'*. Then $B'A'C' = BAC$.

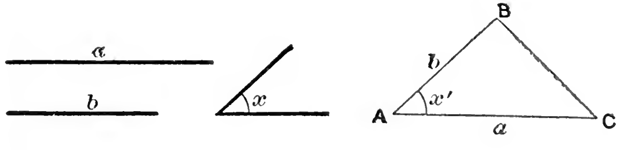
Which geometric axiom and which general axiom apply?

Evidently a visiting-card or any piece of paper with a straight edge will serve the purpose.

Second method (Fig. 2).—With *A* as a centre and any convenient radius describe an arc *xy*. With *A'* as a centre and the same radius describe the indefinite arc *x'z'*. Then take *xy* as a radius, and with *x'* as a centre describe an arc intersecting *x'z'* at *y'*. Join *y'A'*. *y'A'B'* is the angle required.

This cannot be proved until reaching § 89.

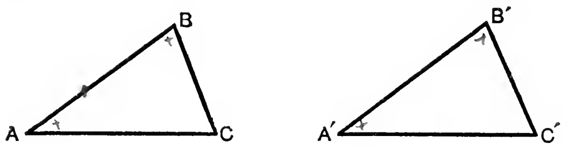
81. CONSTRUCTION. *To form a triangle with two sides and the included angle equal respectively to two lines, a and b, and a given angle, x.*



Lay off $AC = a$. Make $x' = x$ (§ 80). Lay off $AB = b$. Join *BC*. *ABC* is the triangle required, having its two sides and included angle *constructed* as required.

PROPOSITION XXIII. THEOREM

82. *If two triangles have a side and two adjacent angles of one equal to a side and two adjacent angles of the other, the two triangles are equal.*



GIVEN—in the two triangles ABC and $A'B'C'$, $AB = A'B'$, and the angles A and B equal respectively to A' and B'.

TO PROVE the triangles are equal.

Apply ABC to $A'B'C'$ making AB coincide with $A'B'$.

Then AC will fall along $A'C'$, and likewise BC along $B'C'$.

[Since the angles A and B respectively equal A' and B'.]

Hence C must fall somewhere on $A'C'$, and likewise somewhere on $B'C'$.

It must therefore fall on their intersection C'.

And, since the triangles completely coincide, they are equal. Q. E. D.

83. COR. I. *If two triangles have a side and any two angles of one equal respectively to a side and two similarly situated angles of the other, the triangles are equal.*

Hint.—Reduce to the preceding Proposition by § 60.

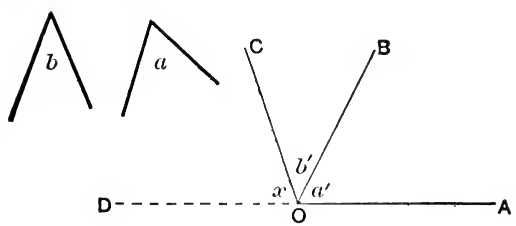
Question.—In how many ways can ABC and $A'B'C'$ have a side and two similarly situated angles equal? Draw two triangles having a side and two angles of each equal but without having the angles similarly situated.

84. *Defs.*—The **hypotenuse** of a right triangle is the side opposite the right angle. The other sides are called the **perpendicular sides.**

85. COR. II. *Two right triangles are equal, if the hypotenuse and an acute angle of one are respectively equal to the hypotenuse and an acute angle of the other.*

86. COR. III. *Two right triangles are equal, if a perpendicular side and an acute angle of one are respectively equal to a perpendicular side and the similarly situated acute angle of the other.*

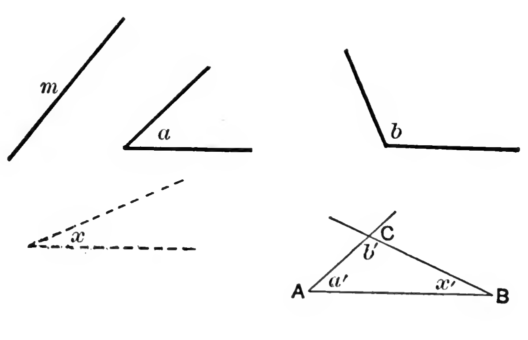
87. CONSTRUCTION. *If two angles of a triangle are equal to given angles a and b, to find the third angle.*



On any line OA construct angle $a' = a$, and on OB at the same vertex O construct $b' = b$. Produce OA to D making the angle x with OC. x is the angle required.

[Prove by § 60.]

88. CONSTRUCTION. *To form a triangle with a side and two angles equal respectively to a given line m and two angles a and b.*



Find (by § 87) x the third angle of the triangle.

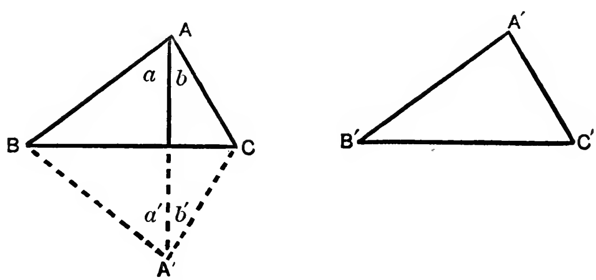
Draw any straight line AB equal to m, and at A and B construct whichever two angles of the three, a, b, x, be required to be adjacent to the given side. If the constructed sides of these angles produced meet, let C be the point of intersection. ABC is the triangle required. For AB equals m by construction, and the angles a' and b' equal a and b by construction or by proof (§ 60).

Discussion.—This problem is impossible if the two given angles are together equal to or greater than two right angles (by § 58).

Question.—Is the problem of § 81 ever impossible?

PROPOSITION XXIV. THEOREM

89. *If two triangles have their three sides respectively equal, they are equal.*



GIVEN—in the triangles ABC and $A'B'C'$, $AB=A'B'$, $BC=B'C'$, and $AC=A'C$.

TO PROVE triangle ABC=triangle $A'B'C'$.

Place $A'B'C'$ so that $B'C'$ shall coincide with its equal BC, but A' shall fall on the side of BC opposite A, and join AA'.

The triangle ABA' has $AB=A'B$, that is, is isosceles. Hyp.

Hence $a=a'$. § 71

[Being base angles of an isosceles triangle.]

Likewise we may prove $b = b'$.

Adding $a + b = a' + b'$. Ax. 2

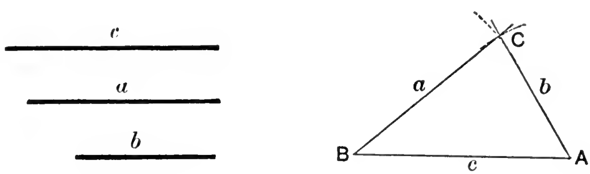
Or angle A = angle A'.

Hence triangle ABC = triangle $A'B'C'$. § 79

[Having two sides and the included angles equal.]

Q. E. D.

90. CONSTRUCTION. *To form a triangle with its three sides equal to given lines a, b, and c.*



Draw AB equal to c. From A as a centre and with b as a radius describe an arc. From B as a centre with a as a radius describe another arc. If these arcs intersect join C, their intersection, with A and B. ABC is the required triangle.

Discussion.—The problem is impossible if one of the given lines is equal to or greater than the sum of the other two.

91. *Exercise.*—By Proposition XXIV. prove that each of the following constructions is correct:

(1.) For erecting a perpendicular, as in § 21, second method.

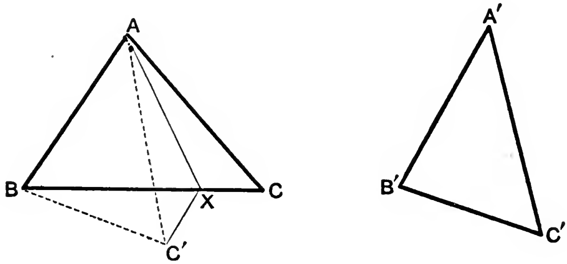
(2.) For making an angle equal to a given angle, as in § 80, second method.

Question.—If two quadrilaterals have their sides equal, each to each, are they necessarily equal?

Question.—In stating Proposition XXIV. does it matter in what order the sides are arranged?

PROPOSITION XXV. THEOREM

92. *If two triangles have two sides of one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, then the third side of the first is greater than the third side of the second.*



GIVEN—two triangles ABC and $A'B'C'$ having $AB=A'B'$ and $AC=A'C'$, but angle $A>$angle A'.

TO PROVE $BC>B'C'$.

Apply $A'B'C'$ to ABC making $A'B'$ coincide with its equal AB.

The angle A' will fall within the angle BAC.

Draw AX bisecting the angle CAC' and meeting BC in X. Join $C'X$.

In the two triangles ACX and $AC'X$

$AC=AC'$, Hyp.

$AX=AX$, Iden.

angle $CAX=$angle $C'AX$. Cons.

Hence triangle $ACX=$triangle $AC'X$. § 79

Hence $XC=XC'$.

Now $BC'<BX+XC'$. § 7

[A straight line is the shortest path between any two of its points.]

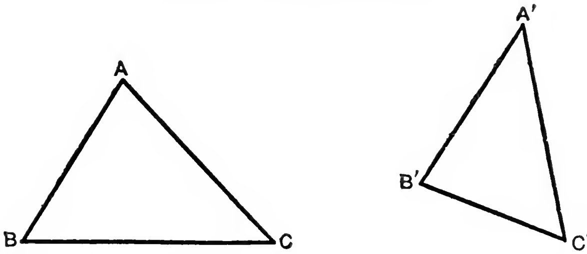
Substituting XC for its equal XC',

$$BC' < BX + XC.$$

Or $$BC' < BC.$$ Q. E. D.

PROPOSITION XXVI. THEOREM

93. *If two triangles have two sides of one equal to two sides of the other but the third side of the first greater than the third side of the second, then the angle opposite the third side of the first is greater than the angle opposite the third side of the second.* [Converse of Proposition XXV.]



GIVEN—in the triangles ABC and $A'B'C'$, $AB = A'B'$ and $AC = A'C'$, but $BC > B'C'$.

TO PROVE angle $A >$ angle A'.

Angle A is either equal to, less than, or greater than angle A'.

If $A = A'$, then would $BC = B'C'$. § 79

[Triangles having two sides and the included angle respectively equal are equal.]

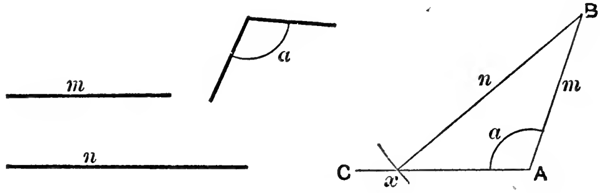
If $A < A'$ then would $BC < B'C'$. § 92

[If two triangles have two sides of one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, then the third side of the first is greater than the third side of the second.]

But both these conclusions contradict the hypothesis.

Therefore $A > A'$. Q. E. D.

94. CONSTRUCTION. *To form a triangle when two sides, m and n, and an angle opposite one of them, a, are given.*



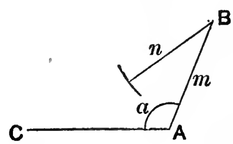
By § 80 construct the given angle a at any vertex A. On one of its sides lay off AB equal to m. From B as a centre with n as a radius draw an arc intersecting the other side at x. ABx is the triangle required.

Discussion.—We may classify two groups of cases.

GROUP I.—*a being greater than an acute angle.*

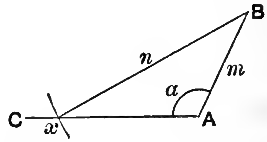
CASE I.—n not longer than m.

No solution.



CASE II.—n longer than m.

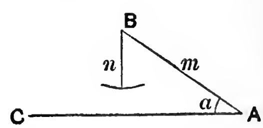
One solution.



GROUP II.—*a being an acute angle.*

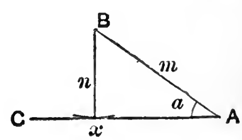
CASE I.—n shorter than the perpendicular from B to AC.

No solution.

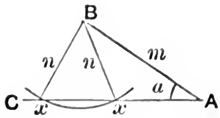


CASE II.—n equal to the perpendicular from B to AC.

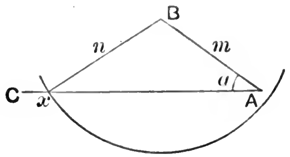
One solution.



CASE III.—n longer than the perpendicular, but shorter than m. Two solutions.

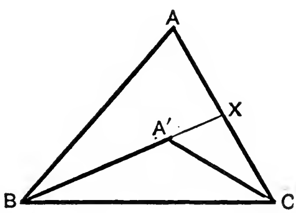


CASE IV.—n not shorter than m. One solution.



PROPOSITION XXVII. THEOREM

95. *If from a point within a triangle two straight lines are drawn to the extremities of one side, their sum will be less than the sum of the other two sides of the triangle.*



GIVEN—the triangle ABC and the lines $A'B$ and $A'C$ drawn from A' to the extremities of BC.

TO PROVE $A'B+A'C<AB+AC$.

Prolong BA' to meet AC at X.

Then $A'C<A'X+XC$. § 7

And also $A'B+A'X<XA+AB$. § 7

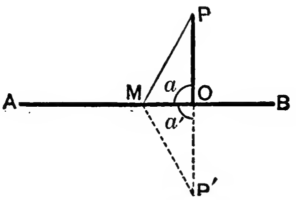
Adding, $A'C+A'B+\underline{A'X}<\underline{A'X}+\overline{XC+XA}+AB$. Ax. 9

Cancel $A'X$ from each side and substitute AC for $XC+XA$.

Then $A'C+A'B<AC+AB$. Q. E. D.

PROPOSITION XXVIII. THEOREM

96. *The perpendicular is the shortest line between a point and a straight line.*



GIVEN—PO the perpendicular from a point P to a straight line AB and PM *any* oblique line from P to AB.

TO PROVE $PO < PM$.

Revolve PMO about AB to form the symmetrical figure $P'MO$. § 32

Then $PO = P'O$ and $PM = P'M$.

Also PO and $P'O$ form a straight line. § 29

[If two adjacent angles (a and a') are together two right angles, their exterior sides form a straight line.]

Now $PP' < PM + MP'$. § 7

Or $2\ PO < 2\ PM$.

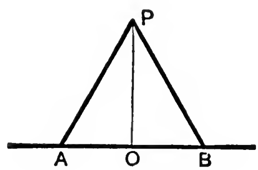
Whence $PO < PM$. Ax. 8

Q. E. D.

97. *Def.*—The "distance" from a point to a straight line means the **shortest** distance, and hence the **perpendicular** distance.

PROPOSITION XXIX. THEOREM

98. *Two oblique lines drawn from the same point in a perpendicular cutting off equal distances from the foot of the perpendicular are equal.*



GIVEN—PO perpendicular to AB, and PA and PB drawn from P cutting off $AO=BO$.

TO PROVE $PA=PB$.

In the *right* triangles POA and POB

$PO=PO$. Iden.

$AO=BO$. Hyp.

Hence triangle POA = triangle POB. § 79

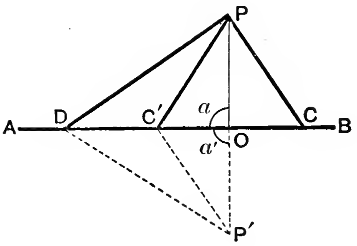
[Having two sides and included angle respectively equal.]

Therefore $PA=PB$.

[Being homologous sides of equal triangles.] Q. E. D.

PROPOSITION XXX. THEOREM

99. *Of two oblique lines drawn from the same point in a perpendicular and cutting off unequal distances from the foot, the more remote is the greater.*



GIVEN PO perpendicular to AB, and OC less than OD.

TO PROVE $PC < PD$.

Take $OC' = OC$ and join PC'.

Then $PC' = PC$. § 98

[Two oblique lines drawn from the same point in a perpendicular cutting off equal distances from the foot of the perpendicular are equal.]

Revolve the figure about AB forming the symmetrical figure $P'DO$.

Then PO and OP' form the same straight line. § 29

[If two adjacent angles (a and a') are together two right angles, their exterior sides form a straight line.]

Now $PC' + P'C' < PD + P'D$. § 95

[If from a point within a triangle, PDP', two straight lines are drawn to the extremities of one side, the sum will be less than the sum of the other two sides of the triangle.]

Substitute PC' for its equal impression $P'C'$, and likewise PD for $P'D$.

Then $2\ PC' < 2\ PD$.

Whence $PC' < PD$. Ax. 8

Substituting PC for PC', $PC < PD$. Q. E. D.

PROPOSITION XXXI. THEOREM

100. *If from a point in a perpendicular to a given straight line two equal oblique lines are drawn, they cut off equal distances from the foot of the perpendicular, and of two unequal oblique lines the greater cuts off the greater distance.*

[Converse of Proposition XXX.]

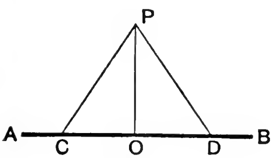


FIG. 1

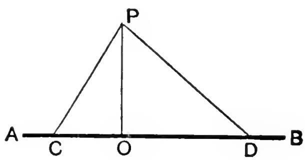


FIG. 2

I. GIVEN PO perpendicular to AB, and $PC = PD$. [Fig. 1.]

TO PROVE $OC = OD$.

OC is either greater than, less than, or equal to OD.

If $OC > OD$, then would $PC > PD$.
If $OC < OD$, then would $PC < PD$. § 99

[Of two oblique lines drawn from the same point in a perpendicular and cutting off unequal distances from the foot of the perpendicular, the more remote is the greater.]

But both these conclusions contradict the hypothesis.

Therefore $OC = OD$. Q. E. D.

II. GIVEN PO perpendicuiar to AB and $PD > PC$. [Fig. 2.]

TO PROVE $OD > OC$.

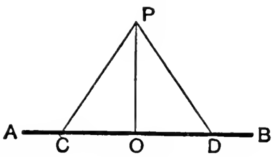


FIG. 1

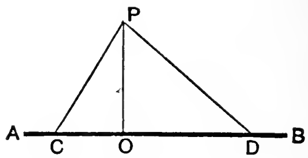


FIG. 2

OD is either equal to, less than, or greater than *OC*.

If $OD = OC$, then would $PD = PC$. § 98

[Two oblique lines drawn from the same point in a perpendicular cutting off equal distances from the foot of the perpendicular are equal.]

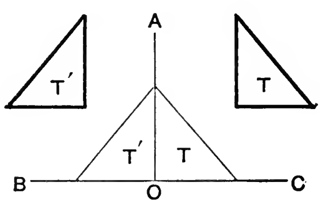
If $OD < OC$, then would $PD < PC$. § 99

[Of two oblique lines drawn from the same point in a perpendicular and cutting off unequal distances from the foot of the perpendicular, the more remote is the greater.]

But both these conclusions contradict the hypothesis.

Therefore $OD > OC$. Q. E. D.

101. Cor. *Two right triangles are equal if they have the hypotenuse and a side of one equal to the hypotenuse and a side of the other.*



Hint.—Draw any two perpendicular lines, *AO* and *BC*, and place the two triangles so that their right angles shall coincide with the right angles at *O* and their equal sides fall along *OA*.

102. *Def.*—A line is the **locus** of all points which satisfy a given condition, if all points in that line satisfy the condition, and no points out of that line satisfy it.

Question—What is the locus of all points three inches from a given point? Prove it.

PROPOSITION XXXII. THEOREM

103. *The locus of all points equally distant from two given points is a straight line bisecting at right angles the line joining the given points.*

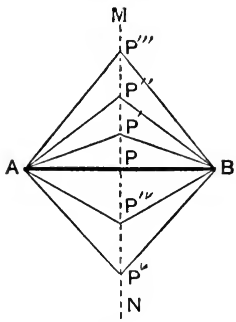


FIG. 1

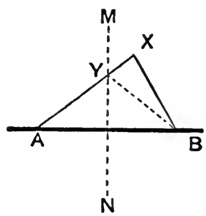


FIG. 2

GIVEN A and B, two fixed points.

TO PROVE—that the locus of all points equally distant from A and B is a straight line MN, perpendicular to AB at its middle point, P.

It is necessary to prove:

I. Every point in MN satisfies the condition of being equally distant from A and B.

II. No point without MN satisfies this condition.

I. (Fig. 1.) Draw MN perpendicular to AB at its middle point, and let P', P'', P''', P^{IV}, etc., be any points in MN.

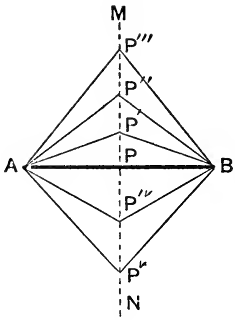


FIG. 1

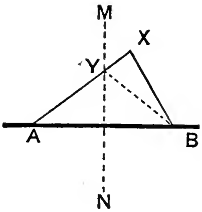


FIG. 2

Then $AP = PB$. Cons.

Hence $P'A = P'B$; $P''A = P''B$; $P'''A = P'''B$, etc. § 98

[Two oblique lines drawn from the same point in a perpendicular cutting off equal distances from the foot of the perpendicular are equal.]

That is, every point in MN is equally distant from A and B.

II. (Fig. 2.) Let X be any point without MN.

Draw XA and XB. One of these lines must cut MN in some point as Y.

Then $XB < XY + YB$. § 7

But $YA = YB$. § 98

Substituting YA for YB, $XB < XY + YA$.

Or $XB < XA$.

Hence every point without MN is unequally distant from A and B. Q. E. D.

104. COR. *Two points equally distant from the extremities of a straight line determine a perpendicular bisector to that line.*

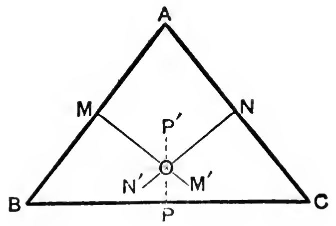
105. *Exercise.*—Show that the following methods of construction were correct:

(1.) Of dropping a perpendicular, as in § 35, second method.

(2.) Of bisecting a straight line, as in § 42, second method.

PROPOSITION XXXIII. THEOREM

106. *The three perpendicular bisectors of the sides of a triangle meet in a common point.*



GIVEN—the triangle ABC and the perpendicular bisectors MM', NN', and PP', of its sides AB, AC, and BC.

TO PROVE—MM', NN', and PP', meet in a common point.

Let O be the intersection of MM' and NN'.

O, being in MM', is equally distant from A and B.
O, being in NN', is equally distant from A and C. } § 103

[The locus of all points equally distant from two fixed points is a straight line bisecting at right angles the line joining the fixed points.]

Hence O is equally distant from B and C.

Hence O lies in PP', the locus of points equally distant from B and C.

Therefore the three perpendicular bisectors meet in a common point. Q. E. D.

107. *Remark.*—This point is the **centre** of the triangle so far as its **vertices** are concerned—that is, it is equally distant from the vertices.

PROPOSITION XXXIV. THEOREM

108. *The bisector of an angle is the locus of all points within the angle equally distant from its sides.*

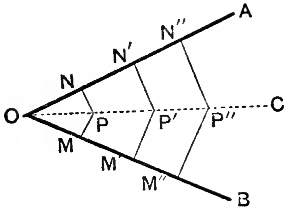


FIG. 1

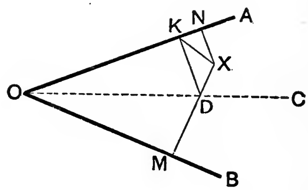


FIG. 2

GIVEN the angle *AOB* and its bisector *OC*.

TO PROVE — *OC* is the locus of all points equally distant from *AO* and *BO*.

It is necessary to prove:

I. That every point in *OC* satisfies the condition of being equally distant from *AO* and *BO*.

II. That any point without *OC* is unequally distant from *AO* and *BO*.

I. (Fig. 1.) Take *P*, any point in *OC*. Draw *PM* and *PN* perpendicular to *OB* and *OA*.

In the right triangles *POM* and *PON*

$OP = OP$, Iden.

angle $POM =$ angle PON. Hyp.

Hence triangle $POM =$ triangle PON. § 85

[Having the hypotenuse and an acute angle respectively equal.]

Therefore $PM = PN$.

[Being homologous sides of equal triangles.]

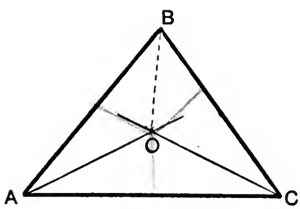
II. (Fig. 2.) Take *X*, any point within the angle, but not in *OC*. Draw *XM* and *XN* perpendicular to *OB* and *OA*.

One of these lines, as XM, must cut OC in some point, as D. Draw DK perpendicular to OA and join XK.

Then	$XN<XK$.	§ 96
And	$XK<XD+DK$.	§ 7
Hence	$XN<XD+DK$.	Ax. 13
But	$DK=DM$.	Part I
	[Since D lies in OC.]	
Therefore	$XN<XD+DM$.	
Or	$XN<XM$.	Q. E. D.

OUTLINE PROOF: $XN<XK<XD+DK=XD+DM=XM$; hence $XN<XM$.

109. COR. *The three bisectors of the angles of a triangle meet in a common point.*



Hint.—Show that the intersection of *two* of the lines must lie on the third as in Proposition XXXIII.

110. *Remark.*—This point is the **centre** of the triangle so far as its **sides** are concerned—that is, it is equally distant from the sides.

111. *Exercise.*—What is the locus of all points equally distant from two intersecting straight lines?

112. *Exercise.*—What is the locus of all points at a given distance from a fixed straight line of indefinite length?

113. *Exercise.*—What is the locus of all points at a given distance from a given line of a definite length?

PARALLELOGRAMS

114. *Defs.*—A **parallelogram** is a quadrilateral whose opposite sides are parallel.

A **rhombus** is a quadrilateral whose sides are all equal.

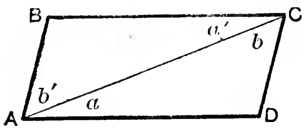
A **rectangle** is a parallelogram whose angles are all right angles.

A **square** is a rectangle whose sides are all equal.

115. *Def.*—A **diagonal** of a quadrilateral is a straight line joining opposite vertices.

PROPOSITION XXXV. THEOREM

116. *A diagonal of a parallelogram divides it into two equal triangles.*



GIVEN the parallelogram $ABCD$ and the diagonal AC.

TO PROVE—that the triangles ABC and ACD are equal.

In the triangles ABC and ACD

$AC = AC$, Iden.

$a = a'$, $b = b'$. § 48

[Being alt.-int. angles of parallel lines.]

Hence triangle ABC = triangle ACD. § 82

[Having a side and two adjacent angles in each respectively equal.]

Q. E. D.

117. COR. I. *In any parallelogram the opposite sides and angles are equal.*

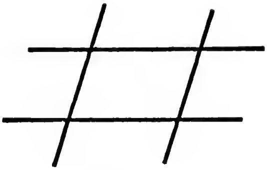


FIG. 1

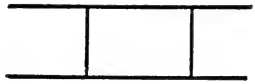


FIG. 2

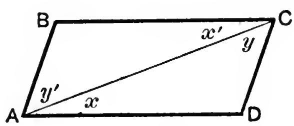
118. COR. II. *Parallels comprehended between parallels are equal.* [Fig. 1.]

119. COR. III. *Parallels are everywhere equally distant.* [Fig. 2.]

Hint.—Apply §§ 33, 36, 118.

PROPOSITION XXXVI. THEOREM

120. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*



GIVEN—any quadrilateral having its opposite sides equal, viz.: $AB=CD$, and $AD=BC$.

TO PROVE the quadrilateral is a parallelogram.

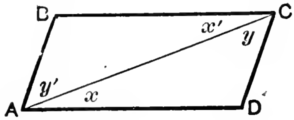
Draw the diagonal AC.

$AC=AC$. Iden.

$AB=CD$.
$AD=BC$. } Hyp.

Hence triangle $ABC=$ triangle ACD. § 89

[Having three sides respectively equal.]



And $x = x'$.

[Being homologous angles of equal triangles.]

Therefore BC is parallel to AD. § 43

[When two straight lines (BC and AD) are cut by a third straight line (AC) making the alternate-interior angles (x and x') equal, the straight lines are parallel.]

In like manner, using y and y', we may prove AB parallel to CD.

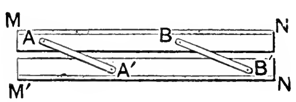
Therefore $ABCD$, having its opposite sides parallel, is a parallelogram. Q. E. D.

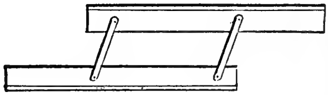
121. A "parallel ruler" is formed by two rulers (MN and $M'N'$), usually of wood pivoted to two metal strips (AA' and BB'), under the following conditions:

(1.) The distances on the rulers between pivots are equal: i. e., $AB = A'B'$.

(2.) The distances on the strips between pivots are equal; i. e., $AA' = BB'$.

(3.) In each ruler the edge is parallel to the line of pivots; i. e., AB is parallel to MN, and $A'B'$ is parallel to $M'N'$.





122. *Exercise.*—Prove: (1.) the quadrilateral whose vertices are the pivots (i. e., the figure $ABB'A'$) is always a parallelogram, whether the ruler be closed or opened.

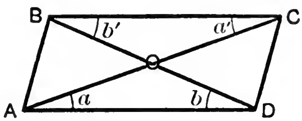
(2.) The edges of the rulers are always parallel (i. e., MN and $M'N'$ are parallel).

123. *Exercise.*—Show how to use the parallel ruler for drawing a straight line through a given point parallel to a given straight line, and prove the method correct.

Extend the method so as to apply even when the point is at a great distance from the line.

PROPOSITION XXXVII. THEOREM

124. *The diagonals of a parallelogram bisect each other.*



GIVEN—a parallelogram $ABCD$ and its diagonals AC and BD intersecting at O.

TO PROVE $AO=OC$ and $OB=OD$.

In the triangles BOC and AOD,

$a=a'$ and $b=b'$. § 48

[Being alt.-int. angles of parallel lines.]

Also $BC=AD$. § 117

[Being opposite sides of a parallelogram.]

Hence triangle BOC = triangle AOD. § 82

[Having a side and two adjacent angles respectively equal.]

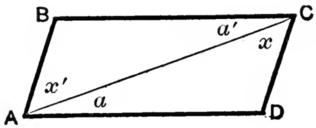
Therefore $AO=OC$ and $BO=OD$.

[Being corresponding sides of equal triangles.] Q. E. D.

125. *Exercise.*—Show that O is a centre of symmetry—that is, that if the figure be turned half way round about O as a pivot (so that OA falls along OC), it will coincide with itself.

PROPOSITION XXXVIII. THEOREM

126. *A quadrilateral which has two of its sides equal and parallel is a parallelogram.*



GIVEN—the quadrilateral $ABCD$ having BC equal and parallel to AD.

TO PROVE $ABCD$ is a parallelogram.

Draw the diagonal AC.

In the triangles ABC and ACD,

$AC = AC$, Iden.

$AD = BC$, Hyp.

angle a = angle a'. § 48

[Being alt.-int. angles.]

Therefore triangle ABC = triangle ACD. § 79

[Having two sides and the included angle respectively equal.]

Hence $x = x'$.

[Being homologous angles of equal triangles.]

Hence AB is parallel to CD. § 43

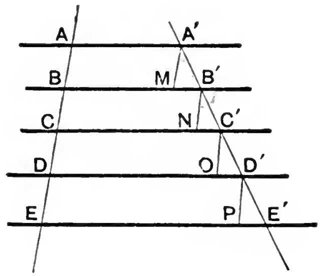
[When two straight lines are cut by a third straight line, making the alt.-int. angles equal, the lines are parallel.]

Therefore $ABCD$ is a parallelogram.

[Having its opposite sides parallel.] Q. E. D.

PROPOSITION XXXIX. THEOREM

127. *If any number of parallels intercept equal parts on one cutting line, they intercept equal parts on every other cutting line.*



GIVEN—AA', BB', CC', DD', EE', any number of parallel lines cutting off the equal parts AB, BC, CD, DE, on AE.

TO PROVE—the parts on any other line $A'E'$ are also equal, viz.: $A'B'$, $B'C'$, $C'D'$, $D'E'$.

Construct parallels to AE through the points A', B', C', D'.

Then $AB = A'M;\ BC = B'N;$ etc. § 118

[Parallels comprehended between parallels are equal.]

But $AB = BC =$ etc. Hyp.

Therefore $A'M = B'N =$ etc. Ax. 1

Also, in the triangles $A'MB'$, $B'NC'$, etc.,

angle A' = angle B' = etc. § 49

[Being corresponding angles of parallels.]

And angle M = angle N = etc. § 51

[Having their sides parallel and in the same order.]

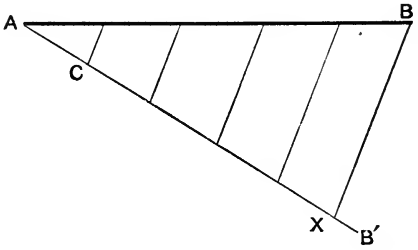
Hence triangle $A'MB'$ = triangle $B'NC'$ = etc. § 83

[Having a side and two angles respectively equal.]

Hence $A'B' = B'C' = C'D' = D'E'$.

[Being homologous sides of equal triangles.] Q. E. D.

128. CONSTRUCTION. *To divide a given line AB into any number of equal parts.*



From A draw any indefinite line AB' and lay off upon it any length AC.

Apply AC the required number of times on AB' and suppose X to be the last point of division. Join XB.

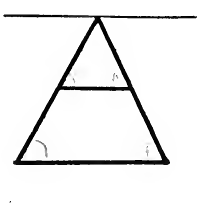
From the various points of division draw parallels to XB.

These parallels will cut AB in the required points of division.

Prove this method correct by Proposition XXXIX.

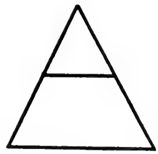
PROBLEMS

129. *Exercise.*—A straight line parallel to the base of a triangle and bisecting one side bisects the other also.



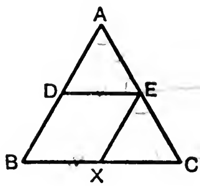
Hint.—Apply § 127.

130. *Exercise*—A straight line joining the middle points of two sides of a triangle is parallel to the third side.



Hint.—Show that this line coincides with a line drawn as in § 129.

131. *Exercise.*—A straight line joining the middle points of two sides of a triangle equals half the third side.

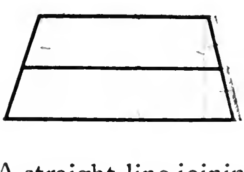


Hint.—Prove $DE=BX$, and $DE=XC$.

132. *Defs.*—A **trapezoid** is a quadrilateral, two of whose sides are parallel.

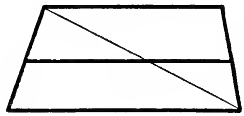
The parallel sides are called the **bases**.

133. *Exercise.*—A straight line parallel to the bases of a trapezoid and bisecting one of the remaining sides bisects the other also.



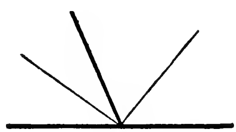
134. *Exercise.*—A straight line joining the middle points of the two non-parallel sides of a trapezoid is parallel to the bases.

135. *Exercise.*—A straight line joining the middle points of the two non-parallel sides of a trapezoid equals half the sum of the bases.



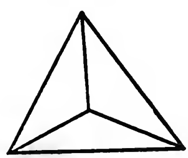
Hint.—Draw a diagonal and apply § 131.

136. *Exercise.*—The bisectors of two supplementary-adjacent angles are perpendicular.



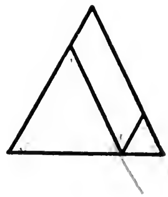
137. *Exercise.*—Any side of a triangle is greater than the difference of the other two.

138. *Exercise.*—The sum of the three lines from any point within a triangle to the three vertices is less than the sum of the three sides, but greater than half their sum.

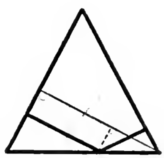


Hint.—Apply §§ 7 and 95.

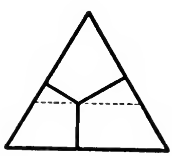
139. *Exercise.*—If from a point in the base of an isosceles triangle parallels to the sides are drawn, a parallelogram is formed, the sum of whose four sides is the same wherever the point is situated (and is equal to the sum of the equal sides).



140. *Exercise.*—If from a point in the base of an isosceles triangle perpendiculars to the sides are drawn, their sum is the same wherever the point is situated (and is equal to the perpendicular from one extremity of the base to the opposite side).

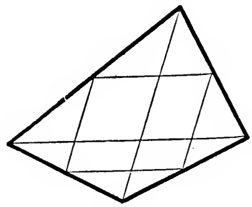


141. *Exercise.*—If from a point within an equilateral triangle perpendiculars to the three sides are drawn, the sum of these lines is the same wherever this point is situated (and is equal to the perpendicular from any vertex to the opposite side).



Hint.—Apply § 140.

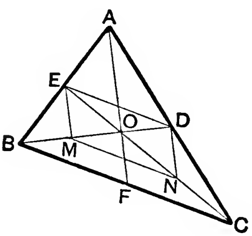
142. *Exercise.*—The straight lines joining the middle points of the adjacent sides of any quadrilateral form a parallelogram.



Hint.—Apply § 130.

143. *Def.*—A **median** of a triangle is a straight line from a vertex to the middle point of the opposite side.

144. *Exercise.*—The three medians of any triangle intersect in a common point which is two-thirds of the distance from each vertex to the middle of the opposite side.



Hint.—*Two* of these lines, CE and BD, meet at some point O.

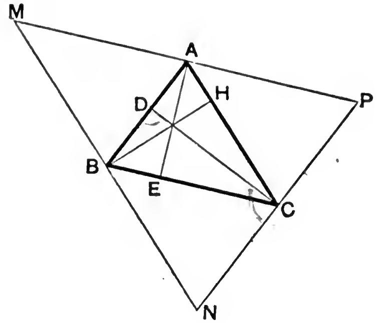
Take M and N, the middle points of BO and CO.

Draw $EDNM$. Prove it is a parallelogram by proving ED and MN each parallel to and equal to half of BC.

Then prove $OE = ON = NC$, and $DO = OM = MB$.

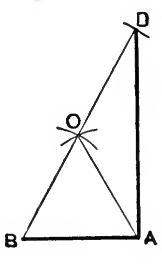
Thus we have proved that one of the medians, as BD, is cut by another, CE, at a point two-thirds of its length from B. We may likewise prove that it is also cut by the third median in the same point. Hence, etc.

145. *Exercise.*—The perpendiculars from the vertices of a triangle to the opposite sides meet in a point.



Hint.—Draw through each vertex a parallel to the opposite side. Prove *AE*, *BH*, and *CD* are perpendicular bisectors of the sides of the new triangle *MNP*, and apply § 106.

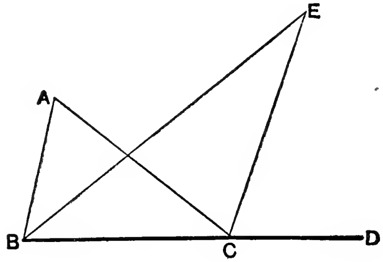
146. *Exercise.*—Prove that the following is a correct method for erecting a perpendicular from a point *A* in a line *AB*.



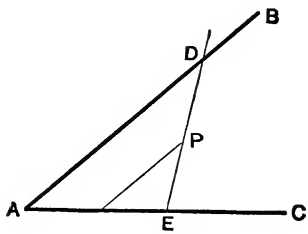
With *A* as a centre describe an arc. With the same radius and any other point, *B*, in the line as a centre, describe a second arc intersecting the first at *O*. With *O* as a centre and the same radius describe a third arc. Join *BO* and produce to meet the third arc at *D*. Join *AD*, the perpendicular required.

Hint.—Of the four right angles of the two triangles, two are at *O*. Show that half the remainder are at *A*.

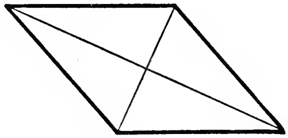
147. *Exercise.*—Given ABC, any triangle. Produce BC. Draw CE bisecting angle ACD, and BE bisecting angle ABC. Prove angle E equals half of angle A.



148. *Exercise.*—Given any angle A and any point P within it. Show a method of drawing a line through P to the sides of the angle which shall be bisected at P.



149. *Exercise.*—The diagonals of a rhombus bisect each other at right angles, and also bisect the angles of the rhombus.



PLANE GEOMETRY

BOOK II

THE CIRCLE

150. * *Def.*—A **circle** is a plane figure bounded by a line, all points of which are equally distant from a point within called the **centre.**

151. * *Defs.*—The line which bounds the circle is called its **circumference.**

An **arc** is any part of a circumference.

152. * *Def.*—Any straight line from the centre to the circumference is a **radius.**

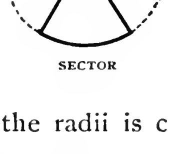
By the definition of a circle all its radii are equal.

153. *Def.*—A **chord** is a straight line having its extremities in the circumference.

154. *Def.*—A **diameter** is a chord through the centre.

All diameters are equal, each being twice a radius.

155. *Defs.*—A **sector** is that portion of a circle bounded by two radii and the intercepted arc.



SECTOR

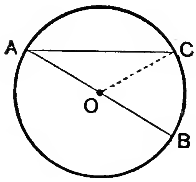
The angle between the radii is called the **angle of the sector.**

* These definitions are repeated from § 20.

156. *Def.*—**Concentric** circles are circles which have the same centre.

PROPOSITION I. THEOREM

157. *The diameter of a circle is greater than any other chord.*



GIVEN—the circle ABC and AC, any chord not a diameter.

TO PROVE $AC <$ diameter AB.

Draw the radius OC.

$AC < AO + OC$. § 7

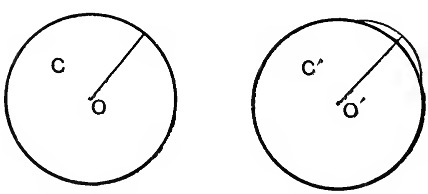
Substitute for OC the equal radius OB.

Then $AC < AO + OB$.

That is $AC < AB$. Q. E. D.

PROPOSITION II. THEOREM

158. *Circles which have equal radii are equal, and if their centres be made to coincide they will coincide throughout; conversely, equal circles have equal radii.*



I. GIVEN—any two circles, C and C' with centres O and O' and equal radii.

TO PROVE the circles C and C' are equal.

Place the circles so that O falls on O'.

Then the circumference of C will coincide with the circumference of C'.

For, if any portion of one fell without the other, its distance from the centre would be greater than the distance of the other. Hence the radii would be unequal, which is contrary to the hypothesis. Ax. 10

Therefore, the circumferences coincide, and the circles coincide and are equal. Q. E. D.

II. CONVERSELY:

GIVEN two equal circles.

TO PROVE their radii equal.

Since the circles are equal they can be made to coincide, and therefore their radii will coincide, and are equal. Q. E. D.

159. COR. I. Hence, *if a circle be turned about its centre as a pivot, its circumference will continue to occupy the same position.*

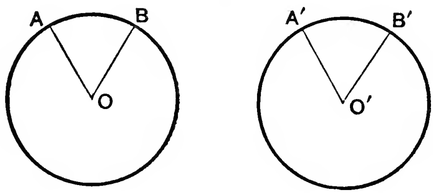
160. COR. II. *The diameter of a circle bisects the circle and the circumference.*

Hint.—Fold over on the diameter as an axis.

161. *Defs.*—The halves into which a diameter divides a circle are called **semicircles**, and the halves into which it divides the circumference are called **semicircumferences.**

PROPOSITION III. THEOREM

162. *In the same circle or equal circles, equal angles at the centre intercept equal arcs; conversely, equal arcs are intercepted by equal angles at the centre.*



I. Given—equal circles and equal angles at their centres, O and O'.

To prove arc AB=arc $A'B'$.

Apply the circles making the angle O coincide with angle O'.

A will coincide with A', and B with B'. § 158

[For $AO=A'O'$, and $OB=O'B'$, being radii of equal circles.]

Then the arc AB will coincide with the arc $A'B'$, and is equal to it. § 150

Q. E. D.

II. Conversely:

Given—equal circles having equal arcs AB and $A'B'$.

To prove—the subtended angles O and O' equal.

Apply the circles making the arc AB coincide with its equal $A'B'$.

Then AO will coincide with $A'O'$, and BO with $B'O'$. Ax. *a*

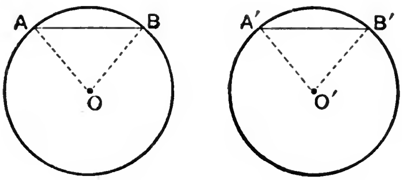
Therefore angles O and O' coincide and are equal. Q. E. D.

163. *Exercise.*—In the same circle or equal circles equal angles at the centre include equal sectors, and conversely.

The proof is analogous to the preceding, requiring "sector" in place of "arc."

PROPOSITION IV. THEOREM

164. *In the same circle or equal circles, equal chords subtend equal arcs; conversely, equal arcs are subtended by equal chords.*



GIVEN—equal circles, O and O', and equal chords, AB and $A'B'$.

TO PROVE arc AB = arc $A'B'$.

Draw the four radii OA, OB, $O'A'$, $O'B'$.

In the triangles AOB and $A'O'B'$

$AB = A'B'$. Hyp.

$AO = A'O'$, and $OB = O'B'$. § 158

[Being radii of equal circles.]

Hence triangle AOB = triangle $A'O'B'$. § 89

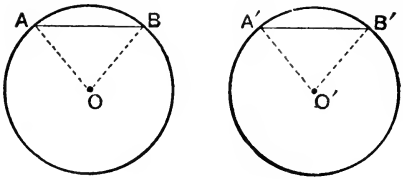
[Having three sides respectively equal.]

Hence angle O = angle O'.

[Being corresponding angles of equal triangles.]

Therefore arc AB = arc $A'B'$. § 162

Q. E. D.



CONVERSELY:

GIVEN—equal circles O and O', and arc AB = arc $A'B'$.

TO PROVE chord AB = chord $A'B'$.

Since the arcs are equal, angle O = angle O'. § 162

And the four radii are equal. § 158

Hence triangle AOB = triangle $A'O'B'$. § 79

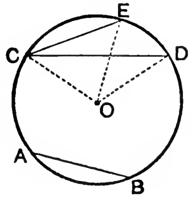
[Having two sides and the included angle respectively equal.]

Therefore chord AB = chord $A'B'$.

[Being corresponding sides of equal triangles.] Q. E. D.

PROPOSITION V. THEOREM

165. *In the same circle or equal circles, if two arcs are unequal and each is less than half a circumference, the greater arc is subtended by the greater chord; conversely, the greater chord subtends the greater arc.*



GIVEN arc CD greater than arc AB.

TO PROVE chord CD greater than chord AB.

Construct upon the greater arc CD an arc CE equal to arc AB.

Then chord CE=chord AB. § 164

Draw the radii OC, OD, OE.

Angle COE is less than angle DOC, being included within it. Ax. 10

Then triangles COE and DOC have two sides (the radii) respectively equal, but the included angles unequal.

Therefore chord CE<chord CD. § 92

Substituting AB for CE,

chord AB<chord CD. Q. E. D.

CONVERSELY:

GIVEN chord CD greater than chord AB.

TO PROVE arc CD greater than arc AB.

As before, construct arc CE equal to arc AB.

Then chord CE=chord AB. § 164

But chord CD>chord AB. Hyp.

Substituting CE for AB,

chord CD>chord CE.

Then the triangles COE and DOC have two sides respectively equal, but the third sides unequal.

Therefore angle COE<angle COD. § 93

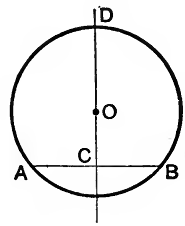
Hence OE, being within the angle DOC, must cut off the arc CE less than the arc CD.

Substituting arc AB for arc CE,

arc AB<arc CD. Q. E. D.

PROPOSITION VI. THEOREM

166. *The perpendicular bisector of a chord passes through the centre of the circle.*



GIVEN—circle *OAB*, chord *AB*, and *CD*, the perpendicular bisector of *AB*.

TO PROVE that *CD* passes through the centre *O*.

CD contains all points equally distant from *A* and *B*. § 103
[Being the locus of such points.]

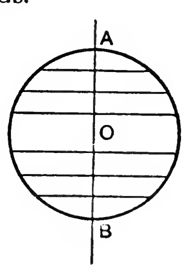
But *O* is such a point, being the centre.

Therefore *CD* contains *O*. Q. E. D.

167. COR. *The diameter perpendicular to a chord bisects it and the subtended arc.*

Hint.—Prove this diameter coincides with the perpendicular bisector. Then draw radii *OA* and *OB*, and apply § 162.

168. *Exercise.*—The locus of the middle points of all chords parallel to a given straight line is a diameter perpendicular to the chords.



The student is cautioned in this, and in exercises about loci in general, not to regard the locus found and proved until he has shown *two* things:

(1.) That every point in the proposed locus satisfies the proposed condition, i. e., is the middle point of one of the parallel chords.

(2.) That every point outside of the proposed locus does not satisfy the required condition, i. e., is not the middle point of any of the parallel chords.

Thus the radius is not the locus, being too small (i. e., requirement 1 would be fulfilled, but not 2); and the diameter produced is not, being too large (i. e., requirement 2 would be fulfilled, but not 1).

Some exercises on loci are more easily proved by showing:

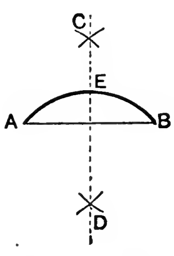
(1.) That every point in the proposed locus satisfies the proposed conditions.

(2.) That every point that satisfies the proposed conditions is in the proposed locus.

The student should show that this method of establishing a locus is equivalent to the previous method.

He may also prove by this method §§ 103 and 108.

169. CONSTRUCTION. *To bisect a given arc.*



GIVEN the arc AEB.

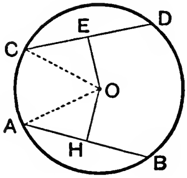
TO CONSTRUCT its bisector.

From A and B as centres, with equal radii greater than a half of AB, describe arcs intersecting at C and D. Draw CD. This line bisects the arc at E.

Hint.—For proof apply § 167.

PROPOSITION VII. THEOREM

170. *In the same circle or equal circles, equal chords are equally distant from the centre; conversely, chords equally distant from the centre are equal.*



GIVEN *CD* and *AB*, equal chords.

TO PROVE—they are at equal distances, *EO* and *HO*, from the centre.

Construct radii *OC* and *OA*.

E and *H* are the middle points of *CD* and *AB*. § 167

In the right triangles *OCE* and *OAH*

$CE = AH$, being halves of equals. Ax. 8

$OC = OA$, being radii.

Hence the triangles are equal. § 101

[Having a side and hypotenuse respectively equal.]

Therefore $OE = OH$. Q. E. D.

CONVERSELY:

GIVEN $OE = OH$.

TO PROVE $CD = AB$.

In the right triangles OCE and OAH

$OE = OH$. Hyp.

$OC = OA$, being radii.

Hence the triangles are equal. § 101

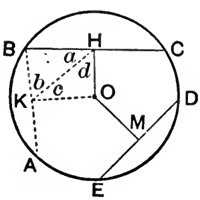
Therefore $CE = AH$.

And $CD = AB$, being doubles of equals. Ax. 7

Q. E. D.

PROPOSITION VIII. THEOREM

171. *In the same circle or equal circles, the less of two chords is at the greater distance from the centre; conversely, the chord at the greater distance from the centre is the less.*



GIVEN chord $ED <$ chord BC.

TO PROVE distance $OM >$ distance OH.

Construct from B chord $BA = ED$.

Then its distance $OK = OM$.

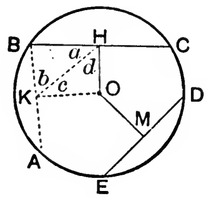
And $AB < BC$.

Join KH.

K and H are the middle points of AB and BC. § 167

Hence $BK < BH$. Ax. 8

[Being halves of unequals.]



Hence angle $a <$ angle b. § 76

[Being opposite unequal sides.]

Subtracting the unequal angles from the equal right angles at H and K,

angle $d >$ angle c. Ax. 6

Therefore $OK > OH$. § 78

[Being opposite unequal angles.]

Substituting OM for OK,

$OM > OH$. Q. E. D.

SUMMARY: $ED < BC$; $BA < BC$; $BK < BH$; $a < b$; $d > c$; $OK > OH$; $OM > OH$.

CONVERSELY:

GIVEN $OM > OH$.

TO PROVE $ED < BC$.

The proof is left to the student.

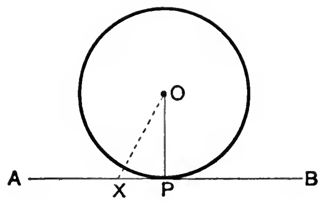
SUMMARY: $OM > OH$; $OK > OH$; $d > c$; $a < b$; $BK < BH$; $BA < BC$; $ED < BC$.

172. *Defs.*—A straight line is **tangent** to a circle if, however far produced, **it meets its circumference in but one point.**

This point is called the **point of tangency.**

PROPOSITION IX. THEOREM

173. *A straight line perpendicular to a radius at its extremity is tangent to the circle; conversely, the tangent at the extremity of a radius is perpendicular to that radius.*



GIVEN—AB perpendicular to the radius OP at its extremity P.

TO PROVE AB is tangent to the circle.

The perpendicular OP is less than any other line OX from O to AB. § 96

[Being the shortest distance from a point to a line.]

Hence, OX being greater than a radius, X lies without the circumference, and P is the only point in AB on the circumference. Therefore AB is tangent to the circle. Q. E. D.

CONVERSELY:

GIVEN AB tangent to the circle at P.

TO PROVE OP perpendicular to AB.

Since AB is touched only at P, any other point in AB, as X, lies without the circumference.

Hence OX is greater than a radius OP.

Therefore OP, being shorter than any other line from O to AB, is perpendicular to AB. § 96

Q. E. D.

174. COR. *A perpendicular to a tangent at the point of tangency passes through the centre of the circle.*

Hint.—Suppose a radius to be drawn to the point of tangency.

175. CONSTRUCTION. *At a point P in the circumference of a circle to draw a tangent to the circle.*

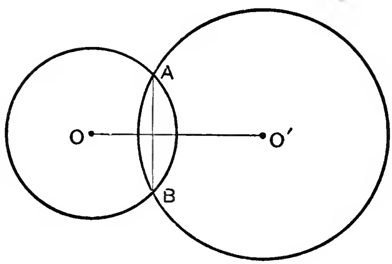
Draw the radius *OP*, and erect *PB* perpendicular to this radius at *P*. By § 173 *PB* is the tangent required.

176. *Exercise.*—The two tangents to a circle from an exterior point are equal.

Hint.—Join the given point and the centre; draw radii to points of tangency.

PROPOSITION X. THEOREM

177. *If two circumferences intersect, the straight line joining their centres bisects their common chord at right angles.*



GIVEN two circumferences intersecting at *A* and *B*.

TO PROVE—*OO′* joining their centres is perpendicular to *AB* at its middle point.

O and *O′* are each equally distant from *A* and *B*. § 150

Therefore *OO′* bisects *AB* at right angles. § 104

[Two points equally distant from the extremities of a straight line determine its perpendicular bisector.] Q. E. D.

MEASUREMENT

178. *Def.*—The **ratio** of one quantity to another of the same kind is the number of times the first contains the second.

Thus the ratio of a yard to a foot is three (3), or more fully $\frac{3}{1}$.

179. *Defs.*—To **measure** a quantity is to find its ratio to another quantity of the same kind. The second quantity is called the **unit of measure**; the ratio is called the **numerical measure** of the first quantity.

Thus we measure the length of a rope by finding the number of metres in it; if it contains 6 metres, we say the *numerical measure* of its length is 6, the metre being the *unit of measure.*

180. *Remark.*—If the length of one rope is 20 metres, and the length of another 5 metres, the ratio of their lengths is the number of times 5 metres is contained in 20 metres—that is, the number of times 5 is contained in 20, which is written $\frac{20}{5}$. We may accordingly restate § 178 as follows:

The ratio of two quantities of the same kind is the ratio of their numerical measures.

181. *Defs.*—Two quantities are **commensurable** if there exists a third quantity which is contained a whole number of times in each.

The third quantity is called the **common measure.**

Thus a yard and a mile are commensurable, each containing a foot a whole number of times, the one 3 times, the other 5280 times. Again, a yard and a rod are commensurable. The common measure is not, however, a foot, as a rod contains a foot $16\frac{1}{2}$ times, which is not a whole number of times. But an inch is a common measure, since the yard contains it 36 times and the rod 198 times.

182. *Def.*—Two quantities are **incommensurable** if no third quantity exists which is contained a whole number of times in each.

Thus it can be proved that the circumference and diameter of a circle are incommensurable; also the side and diagonal of a square.

LIMITS

183. *Def.*—A **constant** quantity is one that maintains the same value throughout the same discussion.

184. *Def.*—A **variable** is a quantity which has different successive values during the same discussion.

185. *Def.*—The **limit** of a variable is a constant *from* which the variable can be made to differ by less than any assigned quantity, but *to* which it can never be made equal.

Thus suppose a point P to move over a line from A to B in such a way that in the first second it passes over half the distance, in the next second half the remaining distance, in the third half the new remainder, and so on.

The variable is the *distance* from A to the moving-point. Its successive values are AP', AP'', AP''', etc. If the length of AB is two inches, the value of the variable is first 1 inch, then $1\frac{1}{2}$, $1\frac{3}{4}$, $1\frac{7}{8}$, etc.

(1.) P will *never* reach B, for there is always half of *some* distance remaining.

(2.) P will approach nearer to B than any quantity we may assign.

Suppose we assign $\frac{1}{1000}$ of an inch. By continually bisecting the remainder we can reduce it to less than $\frac{1}{1000}$ of an inch. Hence the distance from P to A is a variable whose limit is AB, and the distance from P to B is a variable whose limit is zero.

186. THEOREM. *If two variables approaching limits are always equal, their limits are also equal.*

For two variables that are always equal may be considered as but one variable, and must therefore approach the same limit. Q. E. D.

187. LEMMA. *If a variable x can be made smaller than any assigned quantity, then kx, the product of that variable by any constant k, can also be made smaller than any assigned quantity.*

Suppose we assign a quantity s, no matter how small.

We then choose x, so that $x < \frac{s}{k}$.

Therefore, multiplying, $kx < s$. Ax. 7

Q. E. D.

188. COR. *If a variable x can be made as small as we please, so also can x divided by any constant k.*

For $\frac{x}{k}$ is simply $\left(\frac{1}{k}\right) x$, or the product of x by a constant, which we have just proved can be made as small as we please.

189. THEOREM. *The limit of the product of a constant by a variable is the product of that constant by the limit of the variable.*

GIVEN a variable v approaching a limit V.

TO PROVE—the variable kv approaches the limit kV, where k is any constant.

I. kv can never quite equal kV.

For if $kv = kV$,

then would $v = V$, Ax. 8

which is impossible, since v approaches V as a *limit*.

II. kv can be made to differ from kV by less than any assigned quantity.

For their difference, $kV - kv$, may be written $k(V - v)$.

But $V - v$ can be made as small as we please.

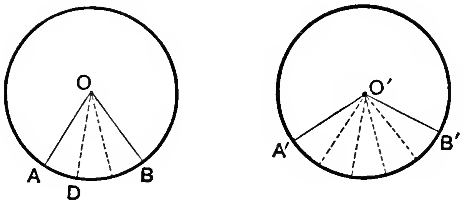
Therefore $k(V - v)$ can be made as small as we please. § 187

Therefore by definition kV is the limit of kv. Q. E. D.

190. COR. *The limit of $\frac{v}{k}$, the quotient of a variable divided by a constant, is $\frac{V}{k}$, the quotient of the limit of the variable divided by the constant k.*

PROPOSITION XI. THEOREM

191. *In the same circle or equal circles, two angles at the centre have the same ratio as their intercepted arcs.*



GIVEN the two equal circles with angles O and O'.

TO PROVE $\frac{\text{angle } O'}{\text{angle } O} = \frac{\text{arc } A'B'}{\text{arc } AB}$.

CASE I. *When the arcs are commensurable.*

Suppose AD is the common measure of the arcs, and is contained three times in AB and five times in $A'B'$.

Then $\frac{\text{arc } A'B'}{\text{arc } AB} = \frac{5}{3}$. § 180

Draw radii to the several points of division.

The angles O and O' will be subdivided into 3 and 5 parts, *all equal.* § 162

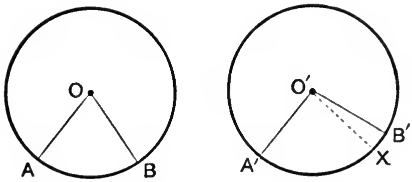
[Being subtended by equal arcs in the same or equal circles.]

Hence $\frac{\text{angle } O'}{\text{angle } O} = \frac{5}{3}$. § 180

Comparing, $\frac{\text{angle } O'}{\text{angle } O} = \frac{\text{arc } A'B'}{\text{arc } AB}$. Ax. 1

Q. E. D.

CASE II. *When the arcs are incommensurable.*



Suppose AB to be divided into any number of equal parts and apply one of these parts to $A'B'$ as a measure as often as it will go.

Since AB and $A'B'$ are incommensurable, there will be a remainder XB' less than one of these parts. § 182

Since AB and $A'X$ are constructed commensurable,

$$\frac{\text{angle } A'O'X}{\text{angle } AOB} = \frac{\text{arc } A'X}{\text{arc } AB}. \qquad \text{Case I}$$

Now suppose the number of parts into which AB is divided to be indefinitely increased.

We can thus make each part as small as we please.

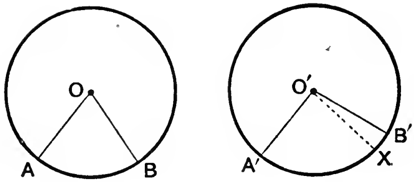
But the remainder, the arc XB', will always be less than one of these parts.

Therefore we can make the arc XB' less than any assigned quantity, though never zero.

Likewise we can make the angle $XO'B'$ less than any assigned quantity, though never zero.

Therefore $A'X$ approaches $A'B'$ as a limit.

Hence $\frac{A'X}{AB}$ approaches $\frac{A'B'}{AB}$ as a limit. § 190



Also angle $A'O'X$ approaches angle $A'O'B'$ as a limit.

Hence $\frac{\text{angle } A'O'X}{\text{angle } AOB}$ approaches $\frac{\text{angle } A'O'B'}{\text{angle } AOB}$ as a limit. § 190

Since the variables $\frac{A'X}{AB}$ and $\frac{\text{angle } A'O'X}{\text{angle } AOB}$ are always equal, so also are their limits.

That is, $$\frac{A'B'}{AB} = \frac{\text{angle } A'O'B'}{\text{angle } AOB}.$$ § 186

Q. E. D.

192. *Exercise.*—In the same circle or equal circles, two sectors have the same ratio as their angles.

The proof is analogous to the preceding, requiring "sector" in place of "arc."

193. *Remark.*—The preceding proposition is often expressed thus:

An angle at the centre *is measured by* its intercepted arc.

This means simply that if the angle is doubled, the intercepted arc will be doubled; if the angle is halved, the intercepted arc will be halved; if the angle is tripled, the intercepted arc will be tripled; and, in general, if the angle is increased or diminished in any ratio, the intercepted arc will be increased or diminished in the same ratio.

194. *Defs.*—A **degree of angle** is one-ninetieth of a right angle.

A **degree of arc** is the arc intercepted by a degree of angle at the centre.

The arc intercepted by a right angle at the centre is called a **quadrant.**

Hence a quadrant contains 90 degrees (90°) of arc, since a right angle contains 90° of angle.

Also, since four right angles contain 360° of angle, and four right angles intercept a complete circumference, a circumference contains 360° of arc.

Hence a quadrant is one-quarter of a circumference.

195. *Remark.*—These definitions suggest a special form of stating Proposition XI., viz.: The ratio of any angle at the centre to a degree of angle equals the ratio of the intercepted arc to the degree of arc, or more briefly: *An angle at the centre contains as many degrees of angle as its intercepted arc contains degrees of arc;* or still again, the numerical measure of an angle at the centre equals the numerical measure of its intercepted arc, the unit of angle being a degree of angle, and the unit of arc being a degree of arc.

The student will be tempted to still further simplify the statement by saying "an angle at the centre *equals* its intercepted arc." This, however, is erroneous, because an angle and an arc are not quantities of the same kind, and can no more be called equal than 23 pounds can be said to be equal to 23 yards.

196. *Def.*—An angle is said to be **inscribed** in a circle, if its vertex lies in the circumference and its sides are chords.

PROPOSITION XII. THEOREM

197. *An inscribed angle is measured by one-half its intercepted arc.**

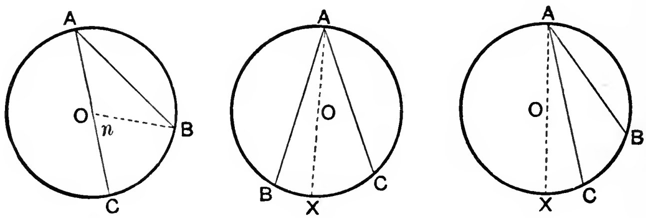


FIG. 1 FIG. 2 FIG. 3

GIVEN the inscribed angle BAC.

TO PROVE—angle BAC is measured by one-half of arc BC.

CASE I. *When one side AC of the angle is a diameter* (Fig. 1).

Draw the radius OB.

$OA = OB$. § 152

[Being radii.]

Hence angle A = angle B. § 71

[Being base angles of an isosceles triangle.]

But angle n = angle A + angle B. § 59

[The exterior angle of a triangle equals the sum of the two opposite interior angles.]

Substituting A for B, $n = 2A$.

But n is measured by arc BC. § 193

Hence half of n, or A, is measured by $\frac{1}{2}$ arc BC. Q. E. D.

* This proposition is first found proved in Euclid (about 300 B.C.), though at least one case, viz., Cor. II. was stated earlier by Thales (about 600 B.C.), the founder of Greek mathematics and philosophy.

CASE II. *When the centre O is within the angle* (Fig. 2).

Construct the diameter AX.

Angle XAC is measured by $\frac{1}{2}$ arc XC. Case I

Angle XAB is measured by $\frac{1}{2}$ arc XB. Case I

Adding, angle BAC is measured by $\frac{1}{2}$ arc $XC+\frac{1}{2}$ arc XB. Ax. 2

Or by $\frac{1}{2}$(arc $XC+$arc XB).

That is by $\frac{1}{2}$ arc BC.

CASE III. *When the centre is without the angle* (Fig. 3).

Construct the diameter AX.

Angle XAB is measured by $\frac{1}{2}$ arc XB. Case I

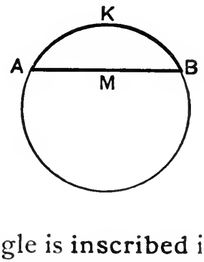
Angle XAC is measured by $\frac{1}{2}$ arc XC. Case I

Subtracting, angle BAC is measured by $\frac{1}{2}$ arc BC. Ax. 3

Q. E. D.

198. *Exercise.*—If the inscribed angle is 37° of angle, how many degrees of arc are there in the intercepted arc? How many in the remainder of the circumference? If the intercepted arc is 17°, how large is the inscribed angle?

199. *Defs.*—A **segment** of a circle is the portion of a circle included between an arc and its chord, as $AKBM$.



200. *Def.*—An angle is **inscribed** in a segment of a circle when its vertex is in the arc of the segment and its sides pass through the extremities of that arc.

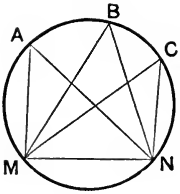


FIG. 1

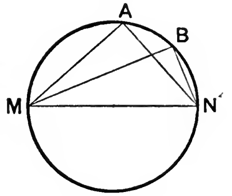


FIG. 2

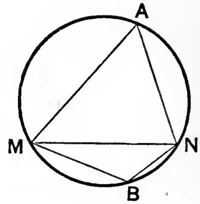


FIG. 3

201. Cor. I. *All angles* (A, B, C, Fig. 1) *inscribed in the same segment are equal.*

For they are measured by one-half the same arc MN.

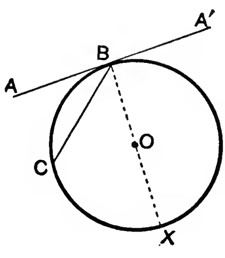
202. Cor. II. *An angle* (A, B, Fig. 2) *inscribed in a semicircle is a right angle.*

203. Cor. III. *An angle* (A, Fig. 3) *inscribed in a segment greater than a semicircle is an acute angle.*

204. Cor. IV. *An angle* (B, Fig. 3) *inscribed in a segment less than a semicircle is an obtuse angle.*

PROPOSITION XIII. THEOREM

205. *An angle formed by a tangent and a chord is measured by one-half its intercepted arc.*



Given—the angle ABC formed by the tangent AB and the chord BC.

To prove—angle ABC is measured by one-half the arc BC.

Construct the diameter BX.

Since a right angle is measured by one-half a semicircumference,

angle ABX is measured by $\frac{1}{2}$ arc BCX.

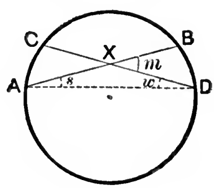
But angle CBX is measured by $\frac{1}{2}$ arc CX. § 197

Subtracting, angle ABC is measured by $\frac{1}{2}$ arc BC. Q. E. D.

206. *Exercise.*—An arc contains $16°$; at its extremities tangents are drawn. What kind of a triangle do they form with the chord, and how large is each angle?

PROPOSITION XIV. THEOREM

207. *The angle between two chords which intersect within the circumference is measured by one-half the sum of its intercepted arc and the arc intercepted by its vertical angle*



GIVEN two intersecting chords AB and CD.

TO PROVE—angle BXD is measured by one-half the sum of the arcs BD and AC.

Join AD.

Now $m = s + w$. § 59

[An exterior angle of a triangle equals the sum of the opposite interior angles.]

But angle s is measured by $\frac{1}{2}$ arc BD. § 197

And angle w is measured by $\frac{1}{2}$ arc AC. § 197

Hence m is measured by $\frac{1}{2}$ (arc BD + arc AC). Ax. 2

Q. E. D.

208. *Exercise.*—One angle of two intersecting chords subtends 30° of arc; its vertical angle subtends 40°. How large is the angle? If an angle of two intersecting chords is 15°, and its intercepted arc is 20°, how large is the opposite arc?

209. *Def.*—A **secant** of a circle is a straight line which cuts the circle.

It is therefore a chord produced.

PROPOSITION XV. THEOREM

210. *The angle between two secants intersecting without the circumference, the angle between a tangent and a secant, and the angle between two tangents, are each measured by one-half the difference of the intercepted arcs.*

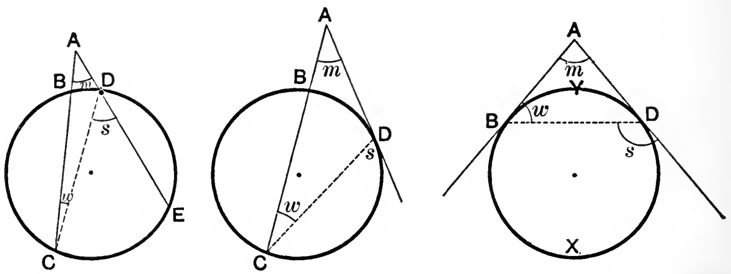


FIG. 1 FIG. 2 FIG. 3

CASE I. *Two secants* (Fig. 1).

GIVEN two secants, AC and AE.

TO PROVE—angle m is measured by $\frac{1}{2}$ (arc CE—arc BD).

Join CD.

Then $m+w=s$. § 59

[An exterior angle of a triangle is equal to the sum of the two opposite interior angles.]

Hence	$m = s - w$.	Ax. 3
But	s is measured by $\frac{1}{2}$ arc CE.	§ 197
And	w is measured by $\frac{1}{2}$ arc BD.	§ 197
Hence	m is measured by $\frac{1}{2}$ (arc CE — arc BD).	Ax. 3

Q. E. D.

CASE II. *A tangent and a secant* (Fig. 2).

GIVEN tangent AD and secant AC.

TO PROVE m is measured by $\frac{1}{2}$ (arc DC — arc BD).

Join CD.

	$m = s - w$.	§ 59
	s is measured by $\frac{1}{2}$ arc DC.	§ 205
	w is measured by $\frac{1}{2}$ arc BD.	§ 197
Hence	m is measured by $\frac{1}{2}$ (arc DC — arc BD).	Ax. 3

Q. E. D.

CASE III. *Two tangents* (Fig. 3).

	$m = s - w$.	§ 59
	s is measured by $\frac{1}{2}$ arc BXD.	§ 205
	w is measured by $\frac{1}{2}$ arc BYD.	§ 205
Hence	m is measured by $\frac{1}{2}$ (arc BXD — arc BYD).	Ax. 3

Q. E. D.

211. *Exercises.*—In Fig. 1, if CE is 50° and BD is 10°, what is m?

In Fig. 1, if m is 16° and BD is 15°, what is CE?

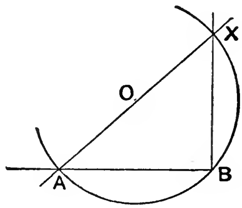
In Fig. 2, if m is 31° and arc DC is 150°, what is arc BD? and what is arc BC?

In Fig. 3, if arc BD is 47°, what is BXD, and what is m?

In Fig. 3, if m is 33°, what are the arcs BXD and BYD?

212. CONSTRUCTION. *At a given point in a straight line to erect a perpendicular.*

[Three methods have been already given, §§ 21, 146.]



GIVEN the straight line AB.

TO CONSTRUCT a perpendicular to AB at B.

With any convenient point O as a centre, and OB as a radius, describe a circumference cutting AB at A and B.

Join OA and produce to meet the circumference at X.

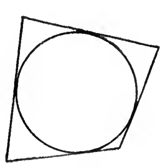
BX is the perpendicular required.

Proof.—Angle ABX is inscribed in a semicircle, and therefore a right angle. § 202

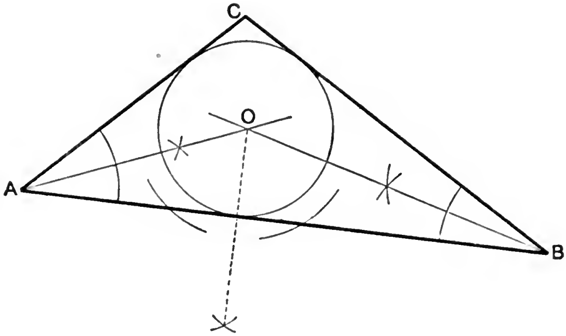
Q. E. D.

213. *Remark.*—The foregoing method is especially convenient when the given point B is near the edge of the paper.

214. *Def.*—A circle is said to be **inscribed** in a polygon, if it be tangent to every side of the polygon. In the same case, the polygon is said to be **circumscribed** about the circle.



215. CONSTRUCTION. *To inscribe a circle in a given triangle.*



GIVEN the triangle ABC.

TO CONSTRUCT an inscribed circle.

Bisect two of the angles, as A and B.

With O, the intersection of these bisectors, as a centre and the distance to any side as a radius, describe a circumference. This gives the circle required.

Proof.—O lies in AO, and is therefore equally distant from AC and AB.

O lies in BO, and is therefore equally distant from BC and BA. § 108

[The bisector of an angle is the locus of points equally distant from its sides.]

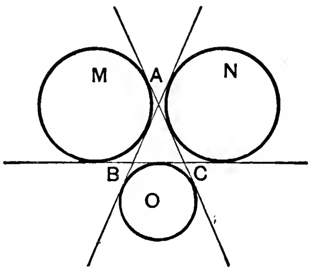
Therefore O is equally distant from *all* sides.

Hence the circle described with O as a centre, and with this distance as a radius, will be tangent to the three sides. § 173

Q. E. D.

216. *Def.*—**Escribed circles** are circles which are tangent to one side of a triangle and the other two sides produced.

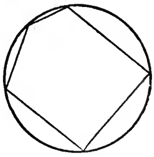
Thus, for the triangle ABC, M, N, and O are escribed circles.



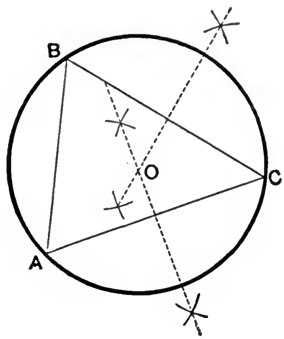
217. *Exercise.*—Construct the three escribed circles of a given triangle.

Hint.—Find centres, as in § 215.

218. *Def.*—A circle is said to be **circumscribed** about a polygon, if the circumference of the circle passes through every vertex of the polygon. In the same case, the polygon is said to be **inscribed** in the circle.



219. CONSTRUCTION. *To circumscribe a circle about a given triangle.*



GIVEN the triangle ABC.

TO CONSTRUCT a circumscribed circle.

Draw the perpendicular bisectors of two of the sides BC and AC.

With O their intersection as a centre, and the distance to any vertex as a radius, describe a circumference.

This gives the circle required.

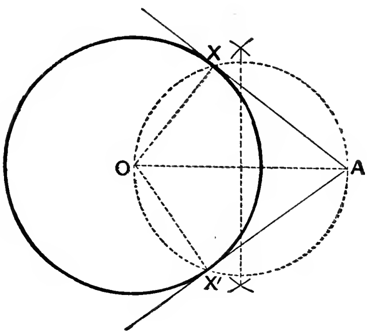
Proof.—O is equally distant from B and C.
O is equally distant from A and C. } § 103

[The perpendicular bisector is the locus of points equally distant from the extremities of a straight line.]

Therefore O is equally distant from *all* vertices, and the circle described as above is the required circle. Q. E. D.

220. *Remark.*—The foregoing construction also enables us to draw a circumference through three points *not in the same straight line* or to find the centre of a given circumference or arc. § 166

221. CONSTRUCTION. *To construct a tangent to a given circle from a given point without.*



GIVEN the circle O and the point A without.

TO CONSTRUCT from A a tangent to the circle.

Upon AO as a diameter construct a circumference intersecting the given circumference at X and X'.

Join AX and AX'.

These lines are the required tangents.

Proof.—Angle AXO is a right angle. § 202

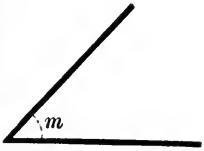
[Being inscribed in a semicircle.]

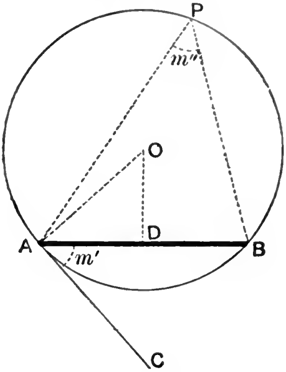
Hence AX is a tangent to the circle O. § 173

[Being perpendicular to a radius at its extremity.]

Likewise AX' is tangent. Q. E. D.

222. CONSTRUCTION. *Upon a given straight line to construct a segment which shall contain a given angle.*





GIVEN the straight line AB and the angle m.

TO CONSTRUCT—a segment upon AB which shall contain an angle equal to m.

At A construct m' equal to m, and having AB as one of its sides. § 80

Draw AO perpendicular to AC, and DO perpendicularly bisecting AB.

With O, the intersection of these two lines, as a centre, and OA or OB as a radius, construct a segment APB. This is the segment required.

Proof.—CA is tangent to the circle. § 173

[Being perpendicular to a radius at its extremity.]

Therefore m' is measured by $\frac{1}{2}$ arc AB § 205

But m'' (any angle inscribed in the segment) is also measured by $\frac{1}{2}$ arc AB. § 197

Therefore $m' = m''$. Ax. 1

But $m = m'$. Cons.

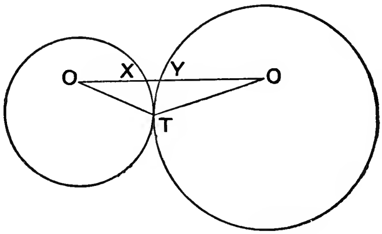
Therefore $m = m''$. Ax. 1

Q. E. D.

PROBLEMS OF DEMONSTRATION

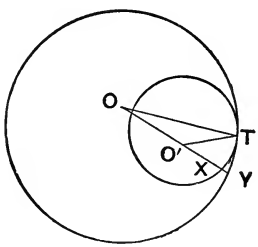
223. *Defs.*—Two circles are **tangent** which touch at but one point. They may be tangent **internally**, so that one circle is within the other; or **externally**, so that each is without the other.

224. *Exercise.*—The straight line joining the centres of two circles tangent externally passes through the point of tangency.



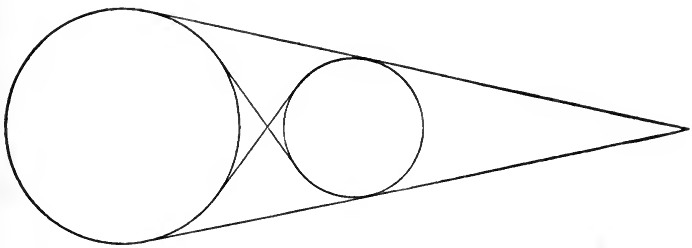
Hint.—Suppose OO' not through T, and prove OO' greater than and also less than the sum of the radii.

225. *Exercise.*—The straight line joining the centres of two circles internally tangent passes through the point of tangency.



Hint.—If not, prove the distance between centres greater than and also less than the difference of the radii.

226. *Defs.*—If each of two circles is entirely without the other, four common tangents can be drawn. Two of these are called external, and two internal. An **external tangent** is one such that the two circles lie on the same side of it; an **internal tangent** is one such that the two circles lie on opposite sides of it.



Question.—In case the two circles are themselves tangent externally, how many common tangents of each kind can be drawn? In case the two circles overlap? In case they are tangent internally? In case one is within the other?

227. *Exercise.*—The two common external tangents to two circles meet the line joining their centres in the same point. Also the two common internal tangents meet the line of centres in the same point.

228. *Exercise.*—The sum of two opposite sides of a quadrilateral circumscribed about a circle is equal to the sum of the other two sides (§ 176).

229. *Exercise.*—The sum of two opposite angles of a quadrilateral inscribed in a circle is equal to the sum of the other two angles, and is equal to two right angles.

230. *Exercise.*—Two circles are tangent externally at A. The line of centres contains A, by § 224. Prove (1) that the perpendicular to the line of centres at A is a common tangent; (2) that it bisects the other two common tangents; and (3) that it is the locus of all points from which tangents drawn to the two circles are equal.

231. *Exercise.*—Find the locus of the middle points of all chords of a given length.

232. *Exercise.*—If a straight line be drawn through the point of contact of two tangent circles forming chords, the radii drawn to the remaining extremities of these chords are parallel. Also, the tangents at these extremities are parallel. What two cases are possible?

PROBLEMS OF CONSTRUCTION

233. *Exercise.*—Draw a straight line tangent to a given circle and parallel to a given straight line.

234. *Exercise.*—Construct a right triangle, given the hypotenuse and an acute angle.

235. *Exercise.*—Construct a right triangle, given the hypotenuse and a side.

236. *Exercise.*—Construct a right triangle, given the hypotenuse and the distance of the hypotenuse from the vertex of the right angle.

237. *Exercise.*—Construct a circle tangent to a given straight line and having its centre in a given point.

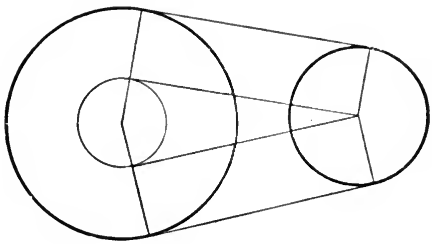
238. *Exercise.*—Construct a circumference having its centre in a given line and passing through two given points.

239. *Exercise.*—Find the locus of the centres of all circles of given radius tangent to a given straight line.

240. *Exercise.*—Construct a circle of given radius tangent to two given straight lines.

241. *Exercise.*—Construct a circle of given radius tangent to two given circles.

242. *Exercise.*—Construct all the common tangents to two given circles.



Hint.—For the external tangents draw a circle with radius equal to the difference of the radii of the given circles and its centre at the centre of the larger circle. Draw tangents to this circle from the centre of the smaller circle.

PLANE GEOMETRY

BOOK III

PROPORTION AND SIMILAR FIGURES

243. *Def.*—A **proportion** is an equality of ratios.

Thus, if the ratio $\frac{A}{B}$ is equal to the ratio $\frac{C}{D}$, then the equality $\frac{A}{B}=\frac{C}{D}$ constitutes a proportion.

This may also be written

$$A:B=C:D, \text{ or } A:B::C:D,$$

and is read, A is to B as C is to D.

244. *Def.*—The four magnitudes A, B, C, D are called the **terms** of the proportion.

245. *Defs.*—The first and last terms are the **extremes,** the second and third, the **means.**

246. *Defs.*—The first and third terms are called the **antecedents,** and the second and fourth, the **consequents.**

247. THEOREM. *If four quantities are in proportion, their numerical measures are in proportion; and conversely.*

GIVEN $$\frac{A}{B}=\frac{C}{D}.$$

TO PROVE: $\frac{a}{b}=\frac{c}{d}$, where a, b, c, d are the numerical measures of A, B, C, D, respectively.

Now $\frac{A}{B}=\frac{a}{b}$ and $\frac{C}{D}=\frac{c}{d}$. § 180

[The ratio of two quantities is equal to the ratio of their numerical measures.]

Whence $\frac{a}{b}=\frac{c}{d}$. Ax. 1

Q. E. D.

CONVERSELY: If $\frac{a}{b}=\frac{c}{d}$, then $\frac{A}{B}=\frac{C}{D}$. This can be proved in like manner.

248. *Remark.*—In order that the preceding theorems shall hold true, A and B must be quantities of the *same kind*, as two straight lines, or two angles, and C and D also of the same kind; *but it is not necessary that A and B shall be of the same kind as C and D.*

249. *Def.*—One variable quantity is said to be **proportional** to another, when any two values of the first have the same ratio as two corresponding values of the second.

Thus, Proposition XI., Book II., may be expressed:

An angle at the centre of a circle is proportional to its intercepted arc.

By this we mean that the ratio of a given angle, as AOB, to some other angle, as $A'O'B'$, is equal to the ratio of the corresponding arcs, AB and $A'B'$.

TRANSFORMATION OF PROPORTIONS

250. THEOREM. *If four numbers are in proportion, the product of the extremes equals the product of the means.*

GIVEN $\frac{a}{b}=\frac{c}{d}$. (1)

TO PROVE $ad=bc$. (2)

Clear (1) of fractions, i. e., multiply both sides by bd, the product of the denominators of (1).

We have $ad=bc$. (2) Ax. 7

Q. E. D.

251. THEOREM. Conversely, *if the product of two numbers equals the product of two others, either pair may be made the extremes and the other pair the means of a proportion.*

GIVEN $ad = bc.$ (2)

TO PROVE $\frac{a}{b} = \frac{c}{d}.$ (1)

Divide both sides of (2) by bd, the product of the denominators of (1).

We have $\frac{a}{b} = \frac{c}{d}.$ (1) Ax. 8

Q. E. D.

Again,

GIVEN $bc = ad.$ (2)

TO PROVE $\frac{b}{a} = \frac{d}{c}.$ (3)

Dividing (2) by ac, the product of the denominators of (3), we obtain (3). Q. E. D.

Question.—By dividing the equation $ad = bc$ by the product of two of the letters, one being from each side, how many proportions in all can be obtained? Write them. If the equation be written $bc = ad$, how many can be obtained, and how do they differ from the former set?

252. *Remark.*—The student has already noticed that the process by which equation (1) was obtained from (2) was the reverse of that by which (2) was obtained from (1). Also it is easy to see that (3) was obtained from (2) by a process the reverse of that by which (2) could have been obtained from (3). Now it is always much easier to see how an equation can be reduced to $ad = bc$ than to see how it can be deduced from $ad = bc$. Since the latter is the reverse of the former, we have the following practical guide for obtaining a required equation from $ad = bc$: First see what processes would be necessary if you wished to reduce the equation to $ad = bc$; reverse these steps in order, and you have the method required.

The preceding rule will be better understood from the following example:

253. If $ad=bc$ (2), prove $\frac{a+b}{b}=\frac{c+d}{d}$. (5)

As it is not at first evident what operations to perform on (2) to obtain (5), let us see what would be necessary in the reverse proof. These operations, as the student will easily see, would be:

Step 1.—Clear (5) of fractions, i. e., multiply both sides by bd.

Step 2.—Cancel bd, i. e., subtract bd from both sides.

By the rule of § 252 we need to reverse these steps, viz.:

First, add bd to both sides of (2).

This gives $ad+bd=bc+bd$. Ax. 2

Secondly, divide both sides by bd.

This gives $\frac{a+b}{b}=\frac{c+d}{d}$. (5) Ax. 8

254. THEOREM. *If four numbers are in proportion, they are also in proportion by inversion.*

GIVEN $\frac{a}{b}=\frac{c}{d}$. (1)

TO PROVE $\frac{b}{a}=\frac{d}{c}$. (3)

OUTLINE PROOF.—Derive from (1) equation (2), or $bc=ad$, and from (2) equation (3) by the rule of § 252.

255. *Exercise.*—Prove § 254 otherwise.

256. THEOREM. *If four numbers are in proportion, they are also in proportion by alternation.*

GIVEN $\frac{a}{b}=\frac{c}{d}$. (1)

TO PROVE $\frac{a}{c}=\frac{b}{d}$. (4)

Hint.—Proceed as in § 254, or multiply each side of (1) by $\frac{b}{c}$.

257. THEOREM. *If four numbers are in proportion, they are also in proportion by composition.*

GIVEN $\frac{a}{b}=\frac{c}{d}$. (1)

TO PROVE $\frac{a+b}{b}=\frac{c+d}{d}$. (5)

Hint.—Proceed as in § 254, or add 1 to each side of equation (1).

258. *Exercise.*—If $\frac{a}{b}=\frac{c}{d}$, prove $\frac{a+b}{a}=\frac{c+d}{c}$.

259. THEOREM. *If four numbers are in proportion, they are also in proportion by division.*

GIVEN $\frac{a}{b}=\frac{c}{d}$. (1)

TO PROVE $\frac{a-b}{b}=\frac{c-d}{d}$. (6)

Hint.—Proceed as in § 254, or subtract 1 from each side of equation (1).

260. *Exercise.*—If $\frac{a}{b}=\frac{c}{d}$, prove $\frac{a-b}{a}=\frac{c-d}{c}$.

261. THEOREM. *If four numbers are in proportion, they are also in proportion by composition and division.*

GIVEN $\frac{a}{b}=\frac{c}{d}$. (1)

TO PROVE $\frac{a+b}{a-b}=\frac{c+d}{c-d}$. (7)

Hint—Divide equation (5) by (6), or proceed as in § 254.

262. THEOREM. *If four numbers are in proportion, equimultiples of the antecedents will be in proportion with equimultiples of the consequents.*

GIVEN $\frac{a}{b}=\frac{c}{d}$. (1)

TO PROVE $\frac{ma}{nb}=\frac{mc}{nd}$. (8)

Hint.—This is proved by multiplying each side of (1) by $\frac{m}{n}$.

263. *Remark.*—The equations so far considered are

$$\frac{a}{b}=\frac{c}{d} \qquad (1)$$

$$ad=bc \qquad (2)$$

$$\frac{b}{a}=\frac{d}{c} \qquad (3)$$

$$\frac{a}{c}=\frac{b}{d} \qquad (4)$$

$$\frac{a+b}{b}=\frac{c+d}{d} \qquad (5)$$

$$\frac{a-b}{b}=\frac{c-d}{d} \qquad (6)$$

$$\frac{a+b}{a-b}=\frac{c+d}{c-d} \qquad (7)$$

$$\frac{ma}{nb}=\frac{mc}{nd}. \qquad (8)$$

The student will see that, if any one of these equations be given, all the others can be obtained. For the given equation can be transformed into (2), and (2) into any other by the method of § 252.

264. *Def.*—A **continued proportion** is an equality of three or more ratios; as

$$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{h}{k}=\text{etc.}$$

265. THEOREM. *In a continued proportion the sum of any number of antecedents is to the sum of the corresponding consequents as any antecedent is to its consequent.*

GIVEN $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{h}{k}=\text{etc.}$

TO PROVE $\frac{a+c+e}{b+d+f}=\frac{a}{b}=\frac{c}{d}=\text{etc.}$

Call each one of the equal ratios $\frac{a}{b}$, $\frac{c}{d}$, etc., r.

Then $\frac{a}{b}=r$, or $a=br$. Ax. 7

$\frac{c}{d}=r$, or $c=dr$.

$\frac{e}{f}=r$, or $e=fr$.

Adding these equations together, we have

$a+c+e=br+dr+fr=r\,(b+d+f)$. Ax. 2

Dividing both sides by $b+d+f$ gives

$\frac{a+c+e}{b+d+f}=r$. Ax. 8

But $r=\frac{a}{b}=\frac{c}{d}=\text{etc.}$

Therefore $\frac{a+c+e}{b+d+f}=\frac{a}{b}=\frac{c}{d}=\text{etc.}$ Ax. 1

Q. E. D.

266. THEOREM. *The products of the corresponding terms of any number of proportions from a proportion.*

GIVEN $\begin{cases} \dfrac{a}{b}=\dfrac{c}{d}, \\ \dfrac{a'}{b'}=\dfrac{c'}{d'}, \\ \dfrac{a''}{b''}=\dfrac{c''}{d''}, \\ \text{etc.} \end{cases}$

TO PROVE $\dfrac{aa'a''}{bb'b''}=\dfrac{cc'c''}{dd'd''}$.

Multiply all the given equations together.

The result is $\dfrac{aa'a''}{bb'b''}=\dfrac{cc'c''}{dd'd''}$. Q. E. D.

267. THEOREM. *If four numbers are in proportion, like powers of these numbers are in proportion.*

GIVEN $\dfrac{a}{b}=\dfrac{c}{d}$.

TO PROVE $\dfrac{a^2}{b^2}=\dfrac{c^2}{d^2}$; $\dfrac{a^3}{b^3}=\dfrac{c^3}{d^3}$; $\dfrac{a^4}{b^4}=\dfrac{c^4}{d^4}$; etc.

This is proved by raising the two sides of the given equation to the required power.

268. *Def.*—The **segments** of a straight line are the parts into which it is divided.

269. *Def.*—Two straight lines are **divided proportionally,** when the ratio of one line to either of its segments is equal to the ratio of the other line to its corresponding segment.

PROPOSITION I. THEOREM

270. *A straight line parallel to one side of a triangle divides the other two sides proportionally.*

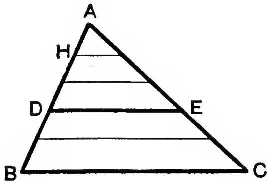


FIG. 1

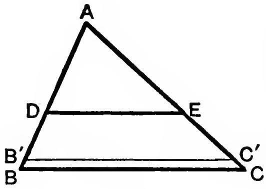


FIG. 2

GIVEN—the straight line DE parallel to the side BC of the triangle ABC.

TO PROVE $$\frac{AB}{AD}=\frac{AC}{AE}.$$

CASE I.—*When AB and AD are commensurable* (Fig. 1).

Let AH be the unit of measure, and suppose it is contained in AB five times, and in AD three times.

Then $$\frac{AB}{AD}=\frac{5}{3}. \quad (1)$$ § 180

Through the several points of division on AB and AD draw lines parallel to BC.

These lines will divide AC into five equal parts, of which AE contains three. § 127

[If any number of parallels intercept equal parts on one cutting line, they will intercept equal parts on every other cutting line.]

Therefore $$\frac{AC}{AE}=\frac{5}{3}. \quad (2)$$ § 180

Comparing (1) and (2),

$$\frac{AB}{AD}=\frac{AC}{AE}.$$ Ax. 1

Q. E. D.

CASE II. *When AB and AD are incommensurable* (Fig. 2).

Let AD be divided into any number of equal parts, and let one of these parts be applied to AB as a measure.

Since AD and AB are incommensurable, a certain number of these parts will extend from A to B', leaving a remainder BB' less than one of these parts.

Through B' draw $B'C'$ parallel to BC.

Since AD and AB' are commensurable,

$$\frac{AB'}{AD}=\frac{AC'}{AE}. \qquad \text{Case I}$$

Now, suppose the number of divisions of AD to be indefinitely increased.

Then each division, either of AD or of AE, can be made as small as we please.

Hence $B'B$ and $C'C$, being always less than one of these divisions, can be made as small as we please.

Hence $\left.\begin{array}{l} AB' \text{ approaches } AB \text{ as a limit.} \\ AC' \text{ approaches } AC \text{ as a limit.}\end{array}\right\}$ § 185

Hence $\left.\begin{array}{l} \dfrac{AB'}{AD} \text{ approaches } \dfrac{AB}{AD} \text{ as a limit.} \\ \dfrac{AC'}{AE} \text{ approaches } \dfrac{AC}{AE} \text{ as a limit.}\end{array}\right\}$ § 190

But we proved $\dfrac{AB'}{AD}=\dfrac{AC'}{AE}.$

Hence $\dfrac{AB}{AD}=\dfrac{AC}{AE}.$ § 186

Q. E. D.

271. COR. I. $\dfrac{AD}{DB}=\dfrac{AE}{EC}.$

Hint—This is proved by division and inversion.

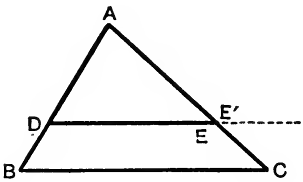
272. COR. II. $\frac{AB}{AC}=\frac{AD}{AE}=\frac{DB}{EC}$.

Hint.—This is proved by alternation.

PROPOSITION II. THEOREM

273. *If a straight line divides two sides of a triangle proportionally, it is parallel to the third side.*

[Converse of Proposition I.]



GIVEN—the straight line DE, in the triangle ABC, so drawn that

$$\frac{AB}{AD}=\frac{AC}{AE}.$$

TO PROVE DE parallel to BC.

From D draw DE' parallel to BC.

Then $\frac{AB}{AD}=\frac{AC}{AE'}$. § 270

[A straight line parallel to one side of a triangle divides the other two sides proportionally.]

But $\frac{AB}{AD}=\frac{AC}{AE}$. Hyp.

Hence $\frac{AC}{AE}=\frac{AC}{AE'}$. Ax. 1

The numerators of these equal fractions being equal, their denominators must also be equal. § 254, Ax. 7

That is, $AE=AE'$.

Therefore E and E' coincide.

Hence DE and DE' coincide. Ax. *a*

But DE' is parallel to BC by construction.

Therefore DE, which coincides with DE', is parallel to BC.

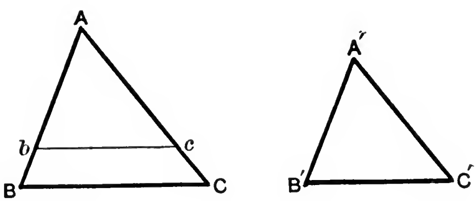
Q. E. D.

274. *Def.*—**Similar polygons** are polygons which have the angles of one equal to the angles of the other, each to each, and the corresponding, or **homologous**, sides proportional.*

As we shall see, if the polygons are triangles, neither of these two conditions can be true without the other; but, if the polygons have four or more sides, either can be true without the other.

PROPOSITION III. THEOREM

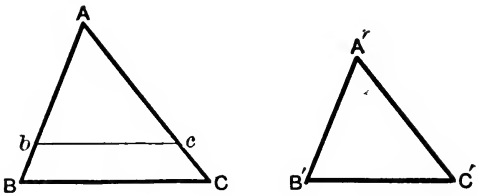
275. *Two triangles which are mutually equiangular are similar.*



GIVEN—in the triangles ABC and $A'B'C'$, the angles A, B, and C, equal respectively to the angles A', B', C'.

TO PROVE the triangle ABC similar to $A'B'C'$.

* There is some evidence that the early Egyptians knew of the properties of similar figures. But the first philosopher who is mentioned as employing them is Thales (600 B.C.). One of his simplest calculations was to find the height of a building by measuring its shadow at that hour of the day when a man's shadow is of the same length as himself.



Apply the triangle $A'B'C'$ to ABC so that the angle A' shall fall on A.

Then the triangle $A'B'C'$ will take the position Abc.

Since the angle Abc (or the angle B') is given equal to B, bc is parallel to BC. § 44

[If two straight lines are cut by a third, so that corresponding angles are equal, the straight lines are parallel.]

Hence $$\frac{AB}{Ab}=\frac{AC}{Ac}.$$ § 270

or $$\frac{AB}{A'B'}=\frac{AC}{A'C'}.$$

By applying the triangle $A'B'C'$ to ABC so that B' shall coincide with its equal B, it may be shown in the same manner that $$\frac{AB}{A'B'}=\frac{BC}{B'C'}.$$

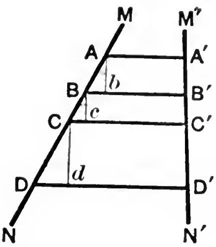
Therefore $$\frac{AB}{A'B'}=\frac{AC}{A'C'}=\frac{BC}{B'C'}.$$ Ax. 1

Hence the homologous sides are proportional and the triangles are similar. § 274

Q. E. D.

276. COR. I. *If two triangles have two angles of one equal to two angles of the other, they are similar.*

277. COR. II. *If two straight lines are cut by a series of parallels, the corresponding segments of the two lines are proportional.*

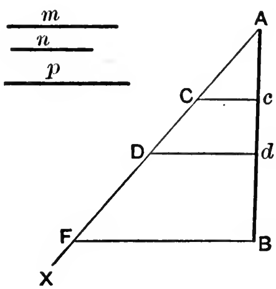


Hint.—Let MN and $M'N'$ be cut by the parallels AA', BB', CC', and DD'.

Draw Ab, Bc, and Cd parallel to $M'N'$.

Prove the triangles ABb, BCc, and CDd similar.

278. CONSTRUCTION. *To divide a given straight line into parts proportional to given straight lines.*



Required.—To divide AB into parts proportional to m, n, and p.

From A draw an indefinite straight line AX, upon which lay off $AC=m$, $CD=n$, and $DF=p$.

Join FB and draw Dd and Cc parallel to FB.

Ac, cd, and dB will then be proportional to m, n, and p. § 277

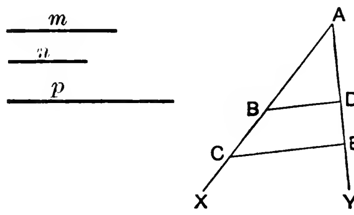
Q. E. F.

279. *Remark.*—If the lines m, n, and p are equal to each other, the line AB will be divided into equal parts. (See also § 127.)

280. *Def.*—A **fourth proportional** to three given quantities is the fourth term of a proportion whose first three terms are the three given quantities taken in order.

281. *Defs.*—When the two means of a proportion are equal, either of them is said to be a **mean proportional** between the other two terms. The fourth term in this case is called a **third proportional** to the other two.

282. CONSTRUCTION. *To find a fourth proportional to three given straight lines.*



Required.—To find a fourth proportional to m, n, and p.

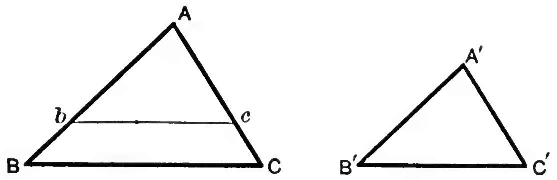
Draw from A the two indefinite lines AX and AY.
Lay off $AB=m$, $AD=n$, and $AC=p$.
Join BD, and through C draw CE parallel to BD.
Then AE will be the fourth proportional.

For $$\frac{AB}{AD}=\frac{AC}{(AE)}.$$ § 272

283. *Remark.*—If n and p are equal, then also AC and AD are equal, and AE is a third proportional to AB and AD.

PROPOSITION IV. THEOREM

284. *Two triangles are similar when their homologous sides are proportional.*



GIVEN—in the two triangles ABC and $A'B'C'$,

$$\frac{AB}{A'B'}=\frac{AC}{A'C'}=\frac{BC}{B'C'}.$$

TO PROVE the triangle ABC similar to $A'B'C'$.

On AB lay off $Ab=A'B'$, and on AC lay off $Ac=A'C'$, and join bc.

Then by substituting Ab and Ac for their equals $A'B'$ and $A'C'$ in the given proportion, we have

$$\frac{AB}{Ab}=\frac{AC}{Ac}.$$

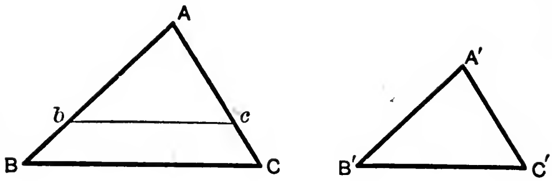
Therefore the line bc is parallel to BC. § 273

[If a straight line divides two sides of a triangle proportionally, it is parallel to the third side.]

And the angle Abc=the angle B, and $Acb=C$. § 49

Hence the triangles ABC and Abc, being mutually equiangular, are similar. § 275

It remains to show that the triangle Abc equals the triangle $A'B'C'$. Since two of their sides are given equal, we only need to show that the third sides bc and $B'C'$ are equal.



Now $\frac{bc}{BC}=\frac{Ab}{AB}=\frac{A'B'}{AB}$. § 274

But $\frac{B'C'}{BC}=\frac{A'B'}{AB}$. Hyp.

Hence $\frac{bc}{BC}=\frac{B'C'}{BC}$. Ax. I

Hence $bc=B'C'$. § 254, Ax. 7

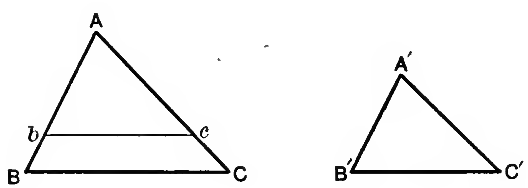
Therefore the triangles Abc and $A'B'C'$ are equal. § 89

But the triangle Abc has been proved similar to ABC.

Hence $A'B'C'$, the equal of Abc, is similar to ABC. Q. E. D.

PROPOSITION V. THEOREM

285. *Two triangles are similar when an angle of the one is equal to an angle of the other, and the sides including these angles are proportional.*



GIVEN—in the triangles ABC and $A'B'C'$, the angle $A=A'$ and

$$\frac{AB}{A'B'}=\frac{AC}{A'C'}.$$

TO PROVE the triangles similar.

Place the triangle $A'B'C'$ on ABC so that the angle A' shall coincide with A, and B' fall at b, and C' at c.

Then $$\frac{AB}{Ab}=\frac{AC}{Ac}.$$ Hyp.

Therefore bc is parallel to BC, § 273

[If a straight line divides two sides of a triangle proportionally, it is parallel to the third side.]

and the angles b and c are equal respectively to B and C. § 49

Hence the triangles ABC and Abc are similar. § 275

[Two triangles which are mutually equiangular are similar.]

But Abc is equal to $A'B'C'$.

Therefore the triangle $A'B'C'$ is also similar to ABC. Q. E. D.

PROPOSITION VI. THEOREM

286. *Two triangles which have their sides parallel each to each, or perpendicular each to each, are similar.*

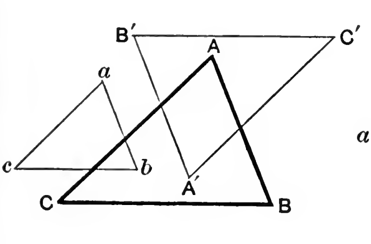


FIG. 1

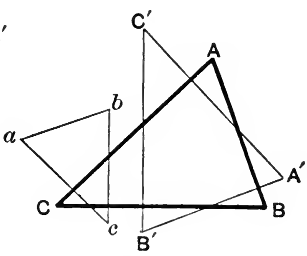


FIG. 2

GIVEN—in the triangles $A'B'C'$ and ABC, that the sides $A'B'$, $A'C'$, and $B'C'$, are respectively parallel to AB, AC, and BC in Fig. 1, and perpendicular in Fig. 2.

TO PROVE the triangles similar.

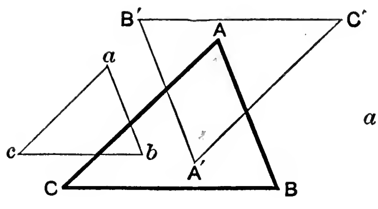


FIG. 1

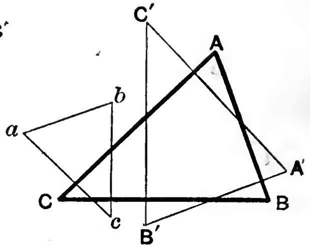


FIG. 2

Since the sides of the two triangles in Fig. 1 are parallel, and in Fig. 2 are perpendicular each to each, the included angles formed by each pair of sides are in both cases either equal or supplementary. §§ 51, 53

Hence, in both cases, we can make three hypotheses, as follows:

1st hypothesis, $A+A'=2$ right angles; $B+B'=2$ right angles; $C+C'=2$ right angles.

2d hypothesis, $A=A'$; $B+B'=2$ right angles; $C+C'=2$ right angles.

3d hypothesis, $A=A'$; $B=B'$; and hence also $C=C'$. § 61

Neither the first nor the second of these hypotheses can be true, for then the sum of the angles of a triangle would be more than two right angles. § 58

Therefore the third is the only one admissible.

Hence the two triangles are similar. Q. E. D.

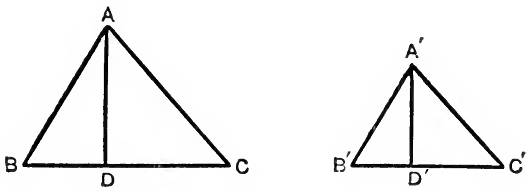
287. *Remark.*—The student will observe that ABC and abc can be proved similar in the same manner.

288. *Remark.*—The homologous sides in the two triangles are any two parallel sides (Fig. 1) or any two perpendicular sides (Fig. 2).

289. *Defs.*—The **base** of a triangle is that side upon which the triangle is supposed to stand. The **altitude** is the perpendicular to the base from the opposite vertex.

PROPOSITION VII. THEOREM

290. *In two similar triangles, corresponding altitudes have the same ratio as any two homologous sides.*



GIVEN—two similar triangles ABC and $A'B'C'$, AD and $A'D'$ being their corresponding altitudes.

TO PROVE $$\frac{AD}{A'D'}=\frac{AB}{A'B'}=\frac{AC}{A'C'}=\frac{BC}{B'C'}.$$

The two right triangles ABD and $A'B'D'$ are similar, since B and B' are equal angles, and ADB and $A'D'B'$ are both right angles. § 276

[If two triangles have two angles of one equal to two angles of the other, they are similar.]

Then $$\frac{AD}{A'D'}=\frac{AB}{A'B'}.$$ § 274

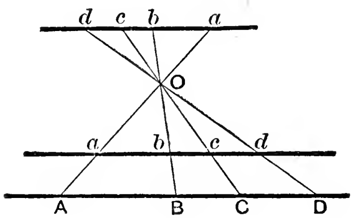
But, since the triangles ABC and $A'B'C'$ are similar, we have $$\frac{AB}{A'B'}=\frac{AC}{A'C'}=\frac{BC}{B'C'}.$$ § 274

Hence $$\frac{AD}{A'D'}=\frac{AB}{A'B'}=\frac{AC}{A'C'}=\frac{BC}{B'C'}.$$ Ax. 1

Q. E. D.

PROPOSITION VIII. THEOREM

291. *If three or more straight lines drawn through a common point intersect two parallels, the corresponding segments of the parallels are proportional.*



GIVEN—the lines OA, OB, OC, OD, drawn through a common point O and intersecting the parallels AD and ad in the points A, B, C, D, and a, b, c, d.

TO PROVE $$\frac{AB}{ab}=\frac{BC}{bc}=\frac{CD}{cd}.$$

Since ad is parallel to AD,
angle Oab = angle OAB, and angle Oba = angle OBA. §§ 48, 49

Therefore the triangle aOb is similar to AOB. § 276

[If two triangles have two angles of one equal to two angles of the other, they are similar.]

In the same way the triangles bOc and cOd are similar respectively to BOC and COD.

Therefore $$\frac{ab}{AB}=\left(\frac{Ob}{OB}\right)=\frac{bc}{BC}=\left(\frac{Oc}{OC}\right)=\frac{cd}{CD}.$$ § 274

Whence $$\frac{ab}{AB}=\frac{bc}{BC}=\frac{cd}{CD}.$$ Ax. 1

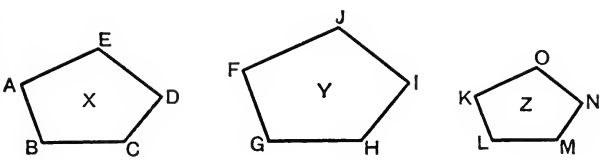
Q. E. D.

292. Cor. If $AB = BC = CD$, then $ab = bc = cd$. Therefore *the lines, drawn from the vertex of a triangle dividing the base into equal parts, divide a parallel to the base into equal parts also.*

293. *Exercise.*—Two men, on opposite sides of a street, walk in opposite directions, and so that a tree between them always hides each from the other. Prove that, if one man walks uniformly, the other must also, and show the connection between the position of the tree and the ratio of their speeds.

PROPOSITION IX. THEOREM

294. *Two polygons similar to a third are similar to each other.*



Given the polygons X and Y, both similar to Z.

To prove that X and Y are similar to each other.

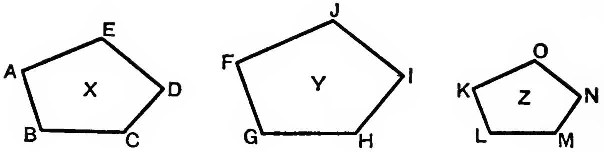
Angles A and F are each equal to K. Hyp.

Therefore they are equal to each other. Ax. 1

In like manner the angles B, C, D, E of X are equal to the corresponding angles of G, H, I, J of Y.

Again $\frac{AB}{KL} = \frac{BC}{LM} = \frac{CD}{MN} = \text{etc.},$

and $\frac{FG}{KL} = \frac{GH}{LM} = \frac{HI}{MN} = \text{etc.}$ § 274



Dividing the first set of equations by the second,

$$\frac{AB}{FG}=\frac{BC}{GH}=\frac{CD}{HI}=\text{etc.}$$

Therefore X and Y are similar. § 274

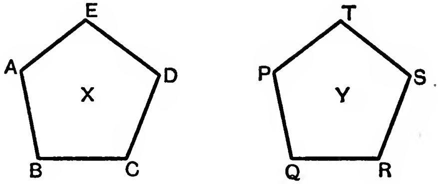
[Having their angles respectively equal and their homologous sides proportional.] Q. E. D.

295. *Def.*—The **ratio of similitude** of any two similar polygons is the ratio of any two homologous sides.

[Thus in § 294 the ratio of AB to FG is the ratio of similitude of X and Y.]

PROPOSITION X. THEOREM

296. *Two similar polygons are equal if their ratio of similitude is unity.*



GIVEN—the similar polygons X and Y, whose ratio of similitude is unity.

TO PROVE X and Y equal.

The angles of X and Y are respectively equal. § 274

Again $$\frac{AB}{PQ} = 1.$$ Hyp.

Therefore $AB = PQ$; likewise $BC = QR$; etc.

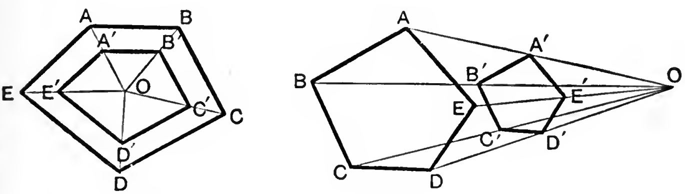
That is, the sides of X and Y are respectively equal.

Hence the polygons, having their corresponding angles and sides respectively equal, can be made to coincide and are equal. Q. E. D.

297. *Defs.*—If the vertices A, B, C, D, etc., of a polygon are joined by straight lines to a point O, and the lines OA, OB, OC, OD, etc., are divided in a given ratio at the points A', B', C', D', etc., the polygon $A'B'C'D'$ etc., is said to be **radially situated** with respect to the polygon $ABCD$, etc.

The ratio of the lines OA' and OA is called the **determining ratio** of the two polygons.

The point O is called the **ray centre.**



In each of the figures the vertices A and A', B and B', C and C', etc., lie on the rays OA, OB, OC, etc., making

$$\frac{OA}{OA'} = \frac{OB}{OB'} = \frac{OC}{OC'} = \text{etc.}$$

The two polygons, $ABCDE$ and $A'B'C'D'E'$, are therefore radially situated.

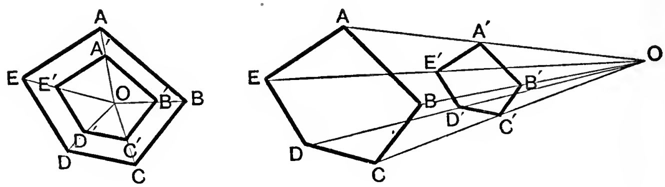
The points A', B', C', D' are **homologous** to the points A, B, C, D respectively.

Straight lines determined by homologous points are **homologous.**

Angles formed by homologous lines are **homologous.**

PROPOSITION XI. THEOREM

298. *Two polygons radially situated are similar and their ratio of similitude is equal to the determining ratio.*



GIVEN—the polygons $ABCDE$ and $A'B'C'D'E'$ radially situated, O being the ray centre.

TO PROVE—they are similar, and that the determining ratio is their ratio of similitude.

AB is parallel to $A'B'$, BC to $B'C'$, etc. § 273

[If a straight line divide two sides of a triangle proportionally, it is parallel to the third side.]

Hence angle $ABC = A'B'C'$, angle $BCD = B'C'D'$, etc. § 51

[Having their sides respectively parallel and in the same right-and-left order.]

Again, triangle OAB is similar to $OA'B'$, OBC to $OB'C'$, etc. § 285

Therefore $$\frac{AB}{A'B'} = \left(\frac{OB}{OB'}\right) = \frac{BC}{B'C'} = \left(\frac{OC}{OC'}\right) = \text{etc.}$$ § 274

Whence $$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \text{etc.}$$ Ax. 1

Since the polygons have their angles respectively equal and their homologous sides proportional, they are similar. § 274

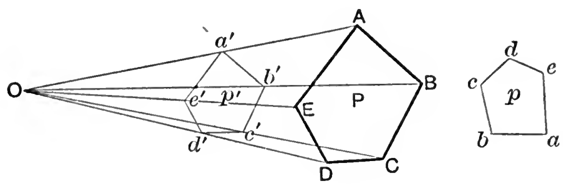
Also, their ratio of similitude $\frac{AB}{A'B'}$ = determining ratio $\frac{OB}{OB'}$.

Q. E. D.

299. *Def.*—The ray centre is also called the **centre of similitude.**

PROPOSITION XII. THEOREM

300. *Any two similar polygons can be radially placed, the determining ratio being equal to the ratio of similitude.*



GIVEN the similar polygons P and p.

TO PROVE—that they can be radially placed, the determining ratio being the ratio of similitude.

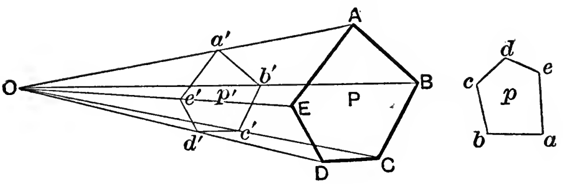
With any point O as ray centre form a polygon p' radially situated with regard to P, having the determining ratio $\frac{Oa'}{OA}$ equal to the ratio of similitude $\frac{ab}{AB}$ of p and P.

Then p' and P will be similar, the ratio of similitude being

$$\frac{a'b'}{AB} = \frac{Oa'}{OA}. \qquad \S\ 298$$

But p and P are given similar, and their ratio of similitude is $\frac{ab}{AB}$.

Therefore p' and p are similar. § 294



Now, since $\frac{a'b'}{AB} = \frac{Oa'}{OA}$ and $\frac{Oa'}{OA} = \frac{ab}{AB}$,

$$\frac{a'b'}{AB} = \frac{ab}{AB}. \qquad \text{Ax. 1}$$

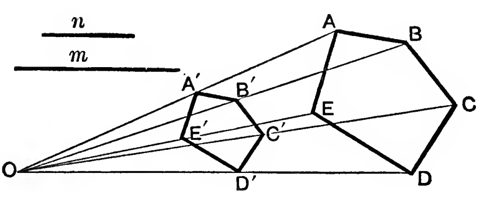
By alternation $$\frac{a'b'}{ab} = \frac{AB}{AB} = 1. \qquad \text{§ 256}$$

That is, the ratio of similitude of p' and p is unity.

Therefore p can be made to coincide with p'. § 296

In other words, P and p can be radially placed, the determining ratio being the ratio of similitude. Q. E. D.

301. CONSTRUCTION. *To draw a polygon similar to a given polygon, having given the ratio of similitude.*



GIVEN the polygon $ABCDE$.

TO CONSTRUCT—similar to $ABCDE$, a polygon $A'B'C'D'E'$, the ratio of similitude being $\frac{m}{n}$.

From any point O draw lines to all the vertices A, B, C, D, E.

Construct OA' a fourth proportional to m, n, and OA. § 282

Likewise find B', C', D', E', so that:

$$\frac{m}{n}=\frac{OA}{OA'}=\frac{OB}{OB'}=\frac{OC}{OC'}=\text{etc.}$$

Then the polygons $ABCDE$ and $A'B'C'D'E'$ are similar, and their ratio of similitude is $\frac{m}{n}$. § 298

Q. E. F.

302. *Exercise.*—To draw a polygon similar to a given polygon, having a given line as a side homologous to a given side of the given polygon.

Hint.—Find the ratio of similitude. Then by § 301 construct a polygon similar to the given polygon having this ratio of similitude. Lastly, upon the given line as a side draw a polygon having its angles and sides equal to those of the second polygon.

303. *Def.*—A **diagonal** of a polygon is a straight line joining two vertices not in the same side.

304. *Exercise.*—In two similar polygons, homologous diagonals have the same ratio as any two homologous sides.

Hint.—Place the polygons in a radial position.

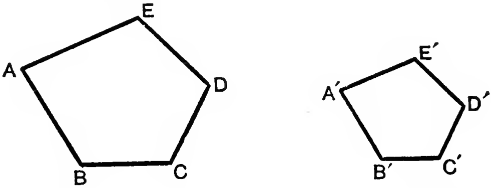
305. *Exercise.*—In two similar polygons, the straight lines joining the middle points of any two pairs of homologous sides are proportional to the sides.

306. *Exercise.*—State and prove a general proposition which includes § 305 as a special case.

307. *Def.*—The **perimeter** of a polygon is the sum of its sides.

PROPOSITION XIII. THEOREM

308. *The perimeters of two similar polygons have the same ratio as any two homologous sides.*



GIVEN—the perimeters P and P' of the two polygons $ABCDE$ and $A'B'C'D'E'$.

TO PROVE $$\frac{P}{P'}=\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CD}{C'D'}=\text{etc.}$$

Since the two polygons are similar, we have

$$\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CD}{C'D'}=\text{etc.} \qquad \S\ 274$$

Then $$\frac{AB+BC+CD+\text{etc.}}{A'B'+B'C'+C'D'+\text{etc.}}=\frac{AB}{A'B'}=\frac{BC}{B'C'}=\text{etc.} \qquad \S\ 265$$

That is, $$\frac{P}{P'}=\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CD}{C'D'}=\text{etc.}$$

Q. E. D.

309. *Remark.*—A pantograph* is a machine for drawing a plane figure similar to a given plane figure.

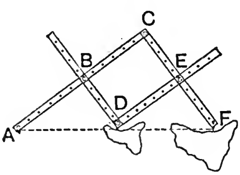


FIG. 1

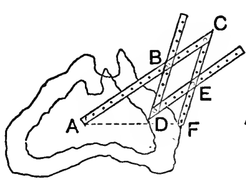


FIG. 2

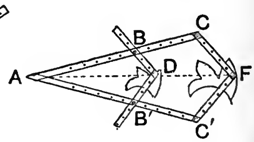


FIG. 3

* The pantograph was invented in 1603 by Christopher Scheiner. It is very useful for enlarging and reducing maps and drawings.

The pantograph, shown in Figs. 1 and 2, consists of four bars, parallel in pairs and jointed at B, C, D, and E. At D and F are pencils and A turns upon a fixed pivot. BD and DE may be so adjusted as to form a parallelogram $BCED$ cutting AC and CF in any required ratio $\frac{AB}{AC}=\frac{CE}{CF}$.

Then (see § 310) D will always be in the same straight line with A and F and the ratio $\frac{AD}{AF}$ will remain constant and equal to the given ratio $\frac{AB}{AC}$.

Hence, if the pencil F traces a given figure, the pencil D will trace a similar figure, the ratio of similitude being the fixed ratio $\frac{AD}{AF}$.

In Fig. 3 the principle is similar; as also in Fig. 4, where the two figures are on opposite sides of A.

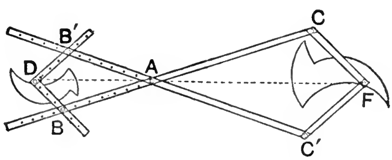


FIG. 4

310. *Exercise.*—Prove the principles stated in § 309, viz., that A, D, F remain always in the same straight line, and that $\frac{AD}{AF}$ remains constant and equal to $\frac{AB}{AC}$.

Hint.—In $\frac{AB}{AC}=\frac{CE}{CF}$ substitute BD for CE and prove the triangles ABD and ACF similar.

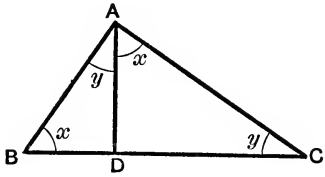
PROPOSITION XIV. THEOREM

311. *In a right triangle, if a perpendicular is drawn from the vertex of the right angle to the hypotenuse:*

I. *The triangles on each side of the perpendicular are similar to the whole triangle and to each other.*

II. *The perpendicular is a mean proportional between the segments of the hypotenuse.*

III. *Each side about the right angle is a mean proportional between the hypotenuse and the adjacent segment.*



GIVEN—the right triangle ABC and the perpendicular AD from the vertex of the right angle A on BC.

I. TO PROVE—the triangles DBA, DAC, and ABC similar to each other.

The right triangles DBA and ABC each have the angle B common; hence they are mutually equiangular. § 61

Also, the right triangles DAC and ABC, having the angle C common, are mutually equiangular. § 61

Hence the three triangles DBA, DAC, and ABC are mutually equiangular.

They are therefore similar. § 275

Q. E. D.

NOTE.—The angles thus proved equal are $B = DAC$, both of which are marked x, and $C = DAB$, both marked y.

II. TO PROVE—AD a mean proportional between DC and BD.

Since the two right triangles DBA and DAC are similar, their homologous sides (that is, the sides opposite equal angles) are proportional. § 274

Hence BD, opposite y in triangle DBA : AD, opposite y in DAC : : AD, opposite x in first : DC, opposite x in second.

That is, AD is a mean proportional between BD and DC.
§ 281

Q. E. D.

III. TO PROVE—AB a mean proportional between BC and BD.

In the similar triangles ABC and DBA.

BC, opposite right angle in the large triangle : BA, opposite right angle in small : : BA, opposite y in first : BD, opposite y in second. § 274

That is, BA is a mean proportional between BC and BD.

In like manner it may be shown that AC is a mean proportional between BC and DC. Q. E. D.

312. COR. I. From II. of the preceding proposition

we have $\overline{AD}^2 = BD \times DC$, (1) § 250

and from III., $\overline{BA}^2 = BC \times BD$, (2)

and $\overline{AC}^2 = BC \times DC$. (3)

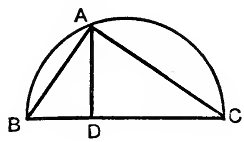
313. COR. II. Dividing (2) by (3)

$$\frac{\overline{BA}^2}{\overline{AC}^2} = \frac{BD}{DC}.$$

Hence, *in a right triangle, the squares of the sides about the right angle are proportional to the segments of the hypotenuse made by a perpendicular let fall from the vertex of the right angle.*

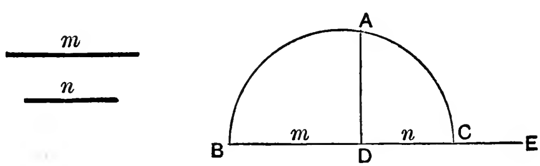
314. *Remark.*—By $\overline{AD}^2$ is understood the square of the numerical measure of *AD*.

315. COR. III. *If from a point A in the circumference of a circle chords AB and AC be drawn to the extremities of a diameter BC, and AD be drawn from A perpendicular to BC,*



AD will be a mean proportional between BD and DC; AB will be a mean proportional between BC and BD; and AC will be a mean proportional between BC and DC.

316. CONSTRUCTION. *To find a mean proportional between two given lines, m and n.*



On the indefinite straight line *BE* lay off $BD = m$ and $DC = n$.

On *BC* as a diameter describe a semicircle.

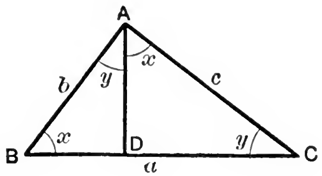
At *D* erect *DA* perpendicular to *BC*, to meet the semicircle.

DA will be a mean proportional between *m* and *n*. § 315.

Q. E. F.

PROPOSITION XV. THEOREM

317. *The square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.**



GIVEN—the right triangle ABC right angled at A, with sides a, b, c.

TO PROVE $b^2+c^2=a^2$.

Draw AD perpendicular to the hypotenuse BC.

Then $\left.\begin{aligned} b^2&=a\times BD \\ c^2&=a\times DC\end{aligned}\right\}$ § 312

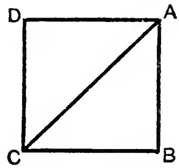
Adding $b^2+c^2=a\times(BD+DC)=a\times a$. Ax. 2

Or $b^2+c^2=a^2$. Q. E. D.

318. COR. I. *The square of either side about the right angle is equal to the difference of the squares of the other two sides.*

* This proposition was first discovered by Pythagoras in the form given in Book IV., Proposition XI. But the Egyptians are supposed to have known as early as 2000 B.C. how to make a right angle by stretching around three pegs a cord measured off into 3, 4, and 5 units. The ancient Hindoos and Chinese also used this method. It is doubtful, however, whether the fact that $3^2+4^2=5^2$ was ever observed by them. It may be noted that essentially this method of forming a right angle is still used by carpenters. Sticks of 6 feet and 8 feet form two sides, and a "ten-foot pole" completes the triangle.

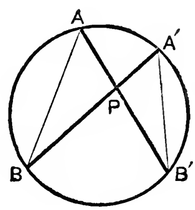
319. COR. II. *The diagonal of a square is equal to the side multiplied by the square root of two.*



OUTLINE PROOF: $AC=\sqrt{\overline{AB}^2+\overline{BC}^2}=\sqrt{2\overline{AB}^2}=AB\sqrt{2}$.

PROPOSITION XVI. THEOREM

320. *If through a fixed point within a circle two chords are drawn, the product of the two segments of one is equal to the product of the two segments of the other.*



GIVEN—P, a fixed point in a circle, and AB' and $A'B$ any two chords drawn through P.

TO PROVE $PA \times PB' = PB \times PA'$.

Join AB and $A'B'$.

In triangles APB, $A'PB'$ angles at P are equal. § 30

[Being vertical.]

Also the angles at A and A' are equal. § 197

[Being inscribed in the same segment.]

Hence the triangles are similar. § 276

Therefore PA, opposite $B : PA'$, opposite $B' :: PB$, opposite $A : PB'$, opposite A'. § 274

Whence $PA \times PB' = PB \times PA'$. § 250

Q. E. D.

PROPOSITION XVII. THEOREM

321. *If from a point without a circle a tangent and a secant be drawn, the tangent is a mean proportional between the whole secant and its external segment.*

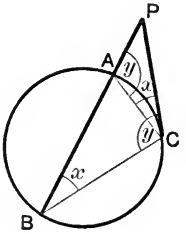


FIG. 1

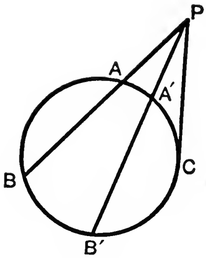


FIG. 2

GIVEN—a fixed point P outside of a circle, PC a tangent, and PB a secant (Fig. 1).

TO PROVE $\frac{PB}{PC} = \frac{PC}{PA}$.

Join AC and BC. The triangles PAC and PCB have the angle at P common, and the angles PCA and PBC (both marked x) equal, each being measured by one-half the arc AC. §§ 197, 205

Therefore the triangles are similar. § 276

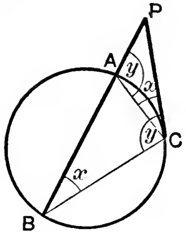


FIG. 1

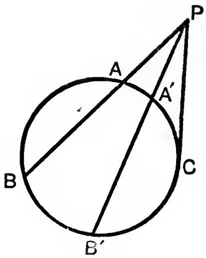


FIG. 2

Hence *PB*, opposite *y* in large triangle : *PC*, opposite *y* in small : : *PC*, opposite *x* in large : *PA*, opposite *x* in small.

Q. E. D.

322. Cor. Hence, in Fig. 2,

$$PB \times PA = \overline{PC}^2,$$

and $$PB' \times PA' = PC^2.$$

Therefore $$PB' \times PA' = PB \times PA.$$ Ax. 1

Hence, *if from a point without a circle two secants be drawn, the product of one secant and its external segment is equal to the product of the other and its external segment.*

323. *Exercise.*—Prove § 322 by drawing *A′B* and *AB′*.

324. *Def.*—The **projection** of a straight line *AB*, upon another straight line *MN*, is the portion of *MN* included between the perpendiculars let fall from the extremities of *AB* upon the line *MN*.

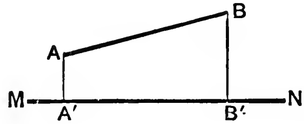


FIG. 1

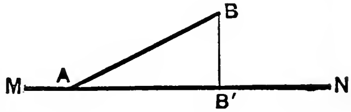


FIG. 2

In Fig. 1 *A′B′* is the projection of *AB*. In Fig. 2, where one extremity of *AB* is on *MN*, *AB′* is the projection.

PROPOSITION XVIII. THEOREM

325. *In any triangle, the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides, minus twice the product of one of these sides and the projection of the other side upon it.*

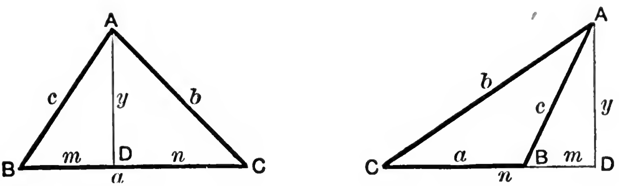


FIG. 1 FIG. 2

GIVEN the triangle ABC and C, an acute angle.

Draw AD perpendicular to CB or CB produced, making CD the projection of AC on CB, and call $AB=c$; $AC=b$; $BC=a$; $AD=y$; $BD=m$; $CD=n$.

TO PROVE $c^2=a^2+b^2-2an$.

In the right triangle ABD.

$$c^2=m^2+y^2. \quad (1)$$ § 317

In Fig. 1, $m=a-n$; and in Fig. 2, $m=n-a$.

In *both* cases $m^2=a^2-2an+n^2$.

Substituting this value in (1),

$$c^2=a^2-2an+n^2+y^2. \quad (2)$$

But in the triangle ACD, $n^2+y^2=b^2$. § 317

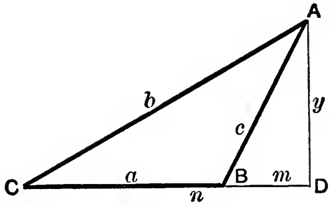
Substituting this value in (2),

$$c^2=a^2+b^2-2an.$$ Q. E. D.

SUMMARY: $c^2=m^2+y^2=a^2-2an+n^2+y^2=a^2-2an+b^2$.

PROPOSITION XIX. THEOREM

326. *In an obtuse-angled triangle the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides, plus twice the product of one of these sides and the projection of the other side upon it.*



GIVEN—the obtuse-angled triangle ABC with B the obtuse angle.

Draw AD perpendicular to CB produced, making BD the projection of AB on CB, and call $AB=c$; $AC=b$; $BC=a$; $AD=y$; $BD=m$; $CD=n$.

TO PROVE $b^2=a^2+c^2+2am$.

In the right triangle ACD

$$b^2=n^2+y^2. \quad (1)$$ § 317

But $n=a+m$.

And $n^2=a^2+2am+m^2$.

Substituting this value of n^2 in (1),

$$b^2=a^2+2am+m^2+y^2. \quad (2)$$

But in the triangle ABD, $m^2+y^2=c^2$. § 317

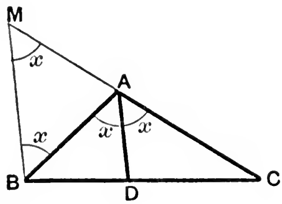
Substituting this value in (2),

$$b^2=a^2+c^2+2am.$$ Q. E. D.

SUMMARY: $b^2=n^2+y^2=a^2+2am+m^2+y^2=a^2+2am+c^2$.

PROPOSITION XX. THEOREM

327. *The bisector of an angle of a triangle divides the opposite side into segments which are proportional to the other two sides.*



GIVEN—in the triangle ABC, AD the bisector of the angle A.

TO PROVE $$\frac{DC}{DB}=\frac{AC}{AB}.$$

Draw BM parallel to AD and meeting AC produced at M.

Then in the triangle BMC, since AD is parallel to BM,

$$\frac{DC}{DB}=\frac{AC}{AM}. \quad (1)$$ § 271

Also, since AD is parallel to MB,

angle $M=DAC$. § 49

[Being corresponding angles of parallel lines.]

And angle $MBA=BAD$. § 48

[Being alt.-int. angles of parallel lines.]

But angle $DAC=BAD$. Hyp.

Therefore angle $M=MBA$. Ax. 1

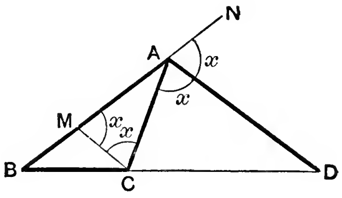
And $AM=AB$. § 77

Substituting in (1), $$\frac{DC}{DB}=\frac{AC}{AB}.$$ Q. E. D.

328. COR. Conversely, *if AD divides BC into two segments which are proportional to the adjacent sides, it bisects the angle BAC.*

PROPOSITION XXI. THEOREM

329. *The bisector of an exterior angle of a triangle meets the opposite side produced in a point whose distances from the extremities of that side are proportional to the other two sides.*



GIVEN—in the triangle *ABC*, *AD* the bisector of the exterior angle *CAN*.

TO PROVE $\frac{DB}{DC}=\frac{AB}{AC}$.

Draw *CM* parallel to *AD*, meeting *AB* at *M*.

Then in the triangle *BAD*, since *CM* is parallel to *AD*,

$$\frac{DB}{DC}=\frac{AB}{AM}. \quad (1)$$ § 272

Also, since *CM* is parallel to *AD*,

angle $AMC=NAD$. § 49

And angle $ACM=CAD$. § 48

But angle $NAD=CAD$. Hyp.

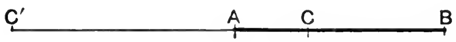
Therefore angle $AMC=ACM$. Ax. 1

And $AM=AC$. § 77

Substituting in (1), $\frac{DB}{DC}=\frac{AB}{AC}$. Q. E. D.

330. COR. Conversely, *if AD meets BC produced so that* $\frac{DB}{DC}=\frac{AB}{AC}$, *then it bisects the angle CAN.*

331. *Defs.*—The line AB is divided **internally** at C, when this point is between the extremities of the line; CA and CB are the segments into which it is divided.



AB is divided **externally** at C', when this point is on the line produced. The segments are $C'A$ and $C'B$.

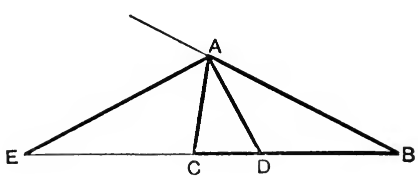
In each case the segments are the distances from the point of division to the extremities of the line. The line is the *sum* of the internal segments, and the *difference* of the external segments.

332. A line is divided **harmonically**, when it is divided internally and externally in the same ratio.

Thus, if $\frac{CA}{CB}=\frac{C'A}{C'B}$, then AB is divided harmonically at C and C'.

333. *Exercise.*—Prove that the bisectors of the interior and exterior angles at one of the vertices of a triangle divide the opposite side harmonically (see figure below).

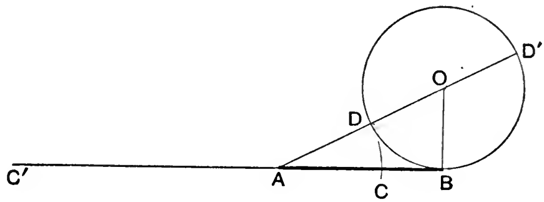
334. *Exercise.*—If AD and AE bisect the angles at A, prove also that ED is divided harmonically at C and B.



Hint.—Alternate the proportion found in § 333.

335. *Def.*—A straight line is divided in **extreme and mean ratio** when one of its segments is a mean proportional between the whole line and the other segment.

336. CONSTRUCTION. *To divide a given straight line in extreme and mean ratio.*



GIVEN the straight line AB.

REQUIRED to divide it in extreme and mean ratio.

At B draw the perpendicular BO equal to one half AB.

With the centre O and radius OB describe a circumference, and draw AO, cutting the circumference in D and D'.

On AB lay off $AC=AD$, and extend BA to C', making $AC'=AD'$.

Then AB is divided in extreme and mean ratio, internally at C, and externally at C'.

I. $$\frac{AD'}{AB}=\frac{AB}{AD}. \quad (1)$$ § 321

By division and inversion

$$\frac{AB}{AD'-AB}=\frac{AD}{AB-AD}. \quad (2)$$ §§ 254, 259

But $AB=2OB=DD'$, and $AD=AC$. Cons.

Therefore,

$AD'-AB=AD'-DD'=AD=AC$, and $AB-AD=BC$.

Substituting these values in (2),

$$\frac{AB}{AC}=\frac{AC}{BC}.$$

Hence AB is divided internally at C in extreme and mean ratio. Q. E. F.

II. By composition and inversion of (1),

$$\frac{AD'}{AD'+AB}=\frac{AB}{AB+AD}. \quad (3) \qquad §§ 254, 257$$

But $AD'=AC'$, and $AB=DD'$.

Therefore $AD'+AB=AC'+AB=BC'$,

And $AB+AD=DD'+AD=AD'=AC'$.

Substituting these values in (3),

we obtain $$\frac{AB}{AC'}=\frac{AC'}{BC'}.$$

Hence AB is divided externally at C' in extreme and mean ratio. Q. E. F.

337. *Remark.*—AC and AC' may be computed in terms of AB as follows:

$$AC=AD=AO-OD=AO-\frac{AB}{2}. \quad (1)$$

Likewise $$AC'=AD'=AO+OD'=AO+\frac{AB}{2}. \quad (2)$$

But $$\overline{AO}^2=\overline{AB}^2+\left(\frac{AB}{2}\right)^2=\overline{AB}^2+\overline{AB}^2.\frac{1}{4}=\overline{AB}^2.\frac{5}{4}. \qquad § 317$$

Whence, extracting the square root,

$$AO=AB.\frac{\sqrt{5}}{2}.$$

Substituting in (1) and (2),

$$AC=AB.\frac{\sqrt{5}}{2}-\frac{AB}{2}=AB.\frac{\sqrt{5}-1}{2}.$$

And $$AC'=AB.\frac{\sqrt{5}}{2}+\frac{AB}{2}=AB.\frac{\sqrt{5}+1}{2}.$$

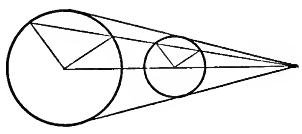
PROBLEMS OF DEMONSTRATION

338. *Exercise.*—The point of intersection of the internal tangents to two circles divides the line of centres internally into parts whose ratio equals the ratio of the radii.

339. *Exercise.*—The point of intersection of the external tangents to two circles divides the line of centres externally into parts whose ratio equals the ratio of the radii.

340. *Exercise.*—The points of intersection of the internal and external tangents to two circles divide the line of centres harmonically.

341. *Exercise.*—If through the centres of two circles two parallel radii are drawn in the same direction, the straight line joining their extremities will pass through the intersection of the external tangents.



342. *Exercise.*—If through the centres of two circles two parallel radii are drawn in opposite directions, the straight line joining their extremities will pass through the intersection of the internal tangents.

343. *Exercise.*—If through the intersection of the external or of the internal tangents to two circles a secant is drawn, the radii to the points of intersection will be parallel in pairs.

344. *Exercise.*—Give methods for drawing the common tangents to two circles depending on §§ 341, 342.

345. *Exercise.*—A triangle ABC is inscribed in a circle to which a second circle is externally tangent at A. If AB and AC are produced till they meet the second circumference at M and N, the triangles ABC and AMN are similar. §§ 205, 275

346. *Exercise*—The perpendiculars from any two vertices of a triangle on the opposite sides are inversely proportional to those sides. § 276

347. *Exercise.*—If two circles are tangent internally, all chords of the greater drawn from the point of contact are divided proportionally by the circumference of the smaller.

Hint.—Apply §§ 202, 225, 276.

348. *Exercise.*—If from P, a point in a circumference, any chords, PA, PB, PC, are drawn, and these chords are cut in a, b, c, respectively, by any straight line parallel to the tangent at P, then $PA \times Pa = PB \times Pb = PC \times Pc$.

Hint.—Let one chord pass through centre. Join its extremity to any other chord and apply §§ 202, 276.

349. *Exercise.*—On a common base AB are two triangles, ABC and ABC', whose vertices C and C' lie in a straight line parallel to AB. If a second parallel to AB cuts AC and BC in M and N, and AC' and BC' in M' and N', then $MN = M'N'$. § 275

350. *Exercise.*—If at the extremities of BC, the hypotenuse of a right triangle ABC, perpendiculars to the hypotenuse are drawn intersecting AB produced in M and AC produced in N, then

$$\frac{AB}{AN} = \frac{AM}{AC}.$$

351. *Exercise.*—The difference of the squares of two sides of any triangle is equal to the difference of the squares of the projections of these sides on the third side. § 317

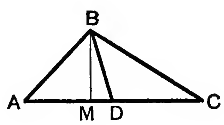
352. *Exercise.*—If from one of the acute angles of a right-angled triangle a straight line be drawn bisecting the opposite side, the square of that line will be less than the square of the hypotenuse by three times the square of half the side bisected.

353. *Exercise.*—If two circles intersect each other, the tangents drawn from any point of their common chord produced are equal. § 321

354. *Exercise.*—If two circles intersect each other, their common chord if produced will bisect their common tangent. § 321

355. *Exercise.*—I. The sum of the squares of two sides of a triangle is equal to twice the square of half the third side, plus twice the square of the median drawn to the third side.

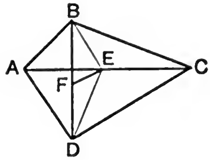
II. The difference of the squares of two sides of a triangle is equal to twice the product of the third side by the projection of the median upon the third side.



Hint.—The median BD divides ABC into two triangles, one acute angled and the other obtuse angled (provided AB and BC are not equal).

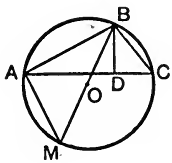
Apply §§ 325, 326.

356. *Exercise.*—In any quadrilateral the sum of the squares of the four sides is equal to the sum of the squares of the diagonals plus four times the square of the line joining the middle points of the diagonals.



Hint.—Apply § 355, I. to the triangles ABC, ADC, and BED, and combine equations thus obtained.

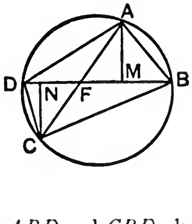
357. *Exercise.*—The product of two sides of a triangle is equal to the product of the diameter of the circumscribed circle and the altitude upon the third side.



Hint.—Let ABC be the triangle. Draw the altitude BD and the diameter BM. Prove the triangles BAM and BDC similar. §§ 201, 202, 276

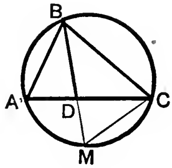
358. *Exercise.*—In an inscribed quadrilateral, $ABCD$, if F is the intersection of the diagonals AC and BD, then

$$\frac{AB \times AD}{CB \times CD} = \frac{AF}{FC}.$$



Hint.—In the triangles ABD and CBD, draw the altitudes AM and CN and apply § 357. Then compare triangles AFM and CFN.

359. *Exercise.*—The product of two sides of a triangle is equal to the square of the bisector of their included angle plus the product of the segments of the third side formed by the bisector.



Hint.—Circumscribe a circle about ABC and produce the bisector to cut the circumference in M. Prove the triangles ABD and MBC similar. Apply § 320.

PROBLEMS OF CONSTRUCTION

360. *Exercise.*—To produce a given straight line MN to a point X, such that $MN : MX = 3 : 7$.

361. *Exercise.*—To construct two straight lines having given their sum and ratio.

362. *Exercise.*—Having given the lesser segment of a straight line divided in extreme and mean ratio, to construct the whole line.

363. *Exercise.*—To construct a triangle having a given perimeter and similar to a given triangle.

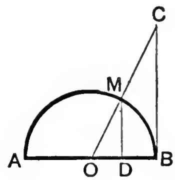
364. *Exercise.*—To construct a right triangle having given an acute angle and the perimeter.

365. *Exercise.*—To divide one side of a given triangle into segments proportional to the other two sides.

366. *Exercise.*—In a given circle to inscribe a triangle similar to a given triangle.

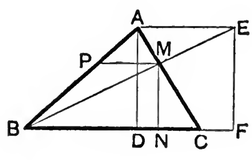
367. *Exercise.*—About a given circle to circumscribe a triangle similar to a given triangle.

368. *Exercise.*—To inscribe a square in a semicircle.



Hint.—At B draw CB equal and perpendicular to the diameter. Join OC cutting the circumference in M, and draw MD parallel to CB. Prove MD the side of the required square by § 275.

369. *Exercise.*—To inscribe a square in a given triangle.

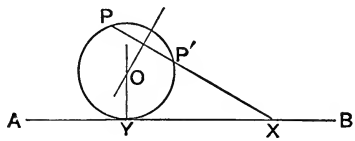


Hint.—On the altitude AD construct the square $ADFE$ and draw BE cutting the side AC at M. From M draw MN and MP parallel to EF and AE respectively. Prove these lines equal and sides of the required square.

370. *Exercise.*—To inscribe in a given triangle a rectangle similar to a given rectangle.

371. *Exercise.*—To inscribe in a given triangle a parallelogram similar to a given parallelogram.

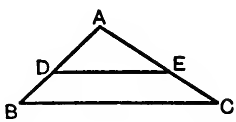
372. *Exercise.*—To construct a circumference which shall pass through two given points and be tangent to a given straight line.



Hint.—Let AB be the given line, P and P' the points. If the straight line PP' is parallel to AB, the solution is simple. If PP' is not parallel to AB, it will cut it at some point X, and the distance from X to Y, the required point of tangency, may be determined by § 321.

PROBLEMS FOR COMPUTATION

373. (1.) In the triangle ABC, DE is drawn parallel to BC. If $\frac{AD}{DB}=\frac{4}{3}$, $BC=56$, and $AE=24$, find AC and DE.



(2.) The sides of a triangle are 3, 5, and 7. In a similar triangle the side homologous to 5 is equal to 65. Find the other two sides of the second triangle.

(3.) The shadow cast upon level ground by a certain church steeple is 27 yds. long, while at the same time that of a vertical rod 5 ft. high is 3 ft. long. Find the height of the steeple.

(4.) The footpaths on the opposite sides of a street are 30 ft. apart. On one of them a bicycle rider is moving uniformly at the rate of 15 miles per hour. If a man on the other side, walking in the opposite direction, so regulates his pace that a tree 5 ft. from his path continually hides him from the rider, does he walk uniformly, and, if so, at what rate does he walk?

(5.) If from the top of a telegraph-pole standing upon the brink of a stream 23 m. wide a wire 30 m. long reaches to the opposite side of the stream, how high is the pole?

(6.) Given the two perpendicular sides of a right triangle equal to 8 and 6 in. respectively to compute the length of the perpendicular from the vertex of the right angle to the hypotenuse.

(7.) If in a right triangle the two perpendicular sides are a and b, compute the altitude upon the hypotenuse.

(8.) If, in the above example, $a = 137.53$ dkm., and $b = 213.19$ m., find the altitude.

(9.) If in a right triangle one of the sides about the right angle is double the other, what is the ratio of the segments of the hypotenuse formed by the altitude upon the hypotenuse?

(10.) There are two telegraph-poles standing upon the same level in a city street, one 59 ft. high, the other 45 ft. high, while between them, and in a straight line with their bases, is a hitching-post 3 ft. high. If the distance from the top of the post to the top of the higher pole is 100 ft., and from the top of the post to that of the lower pole 80 ft., how far apart are the poles?

(11.) If the chord of an arc is 720 ft. and the chord of its half is 369 ft., what is the diameter of the circle?

(12.) A chord of a circle is divided into two segments of 73.162 dcm. and 96.758 dcm. respectively by another chord, one of whose segments is 3.1527 m. What is the length of the second chord?

(13.) If a chord of a circle is cut by another chord into two segments, a and b, and one segment of the second chord is equal to c, find the other segment.

(14.) If from a point without a circle two secants are drawn whose external segments are 8 in. and 7 in., while the internal segment of the latter is 17 in., what is the length of the internal segment of the former?

(15.) From a point without a circle are drawn a tangent and a secant, the secant passing through the centre. If the length of the tangent is a, and the external segment of the secant is b, find the radius of the circle.

(16.) In a triangle whose sides are respectively 25.136 cm., 31.298 cm., and 37.563 cm. in length, find the segments of the longest side formed by the bisector of the opposite angle.

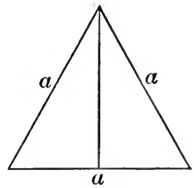
(17.) In a triangle whose sides are a, b, and c, find the segments of the side b formed by the bisector of the opposite angle.

(18.) If the base of an isosceles triangle is 60 cm., and each of its sides is 50 cm., find the length of its altitude in inches.

(19.) If the base of an isosceles triangle is b, and its altitude h, find the sides.

(20.) Find the altitude of an equilateral triangle whose side is 5 in.

(21.) Show that, if a is the side of an equilateral triangle, the altitude is $\frac{1}{2}a\sqrt{3}$.



(22.) Find in feet the side of an equilateral triangle having an altitude of 793.57 m.

(23.) Show that, in a right triangle, one of whose acute angles is 30°, and whose hypotenuse is a, the side opposite 30° is $\frac{1}{2}a$, and the other side is $\frac{1}{2}a\sqrt{3}$.

(24.) One acute angle of a right triangle is 30° and the hypotenuse is 4.3791 cm. Find the other sides.

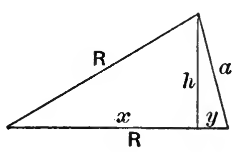
(25.) Find the side of an isosceles right triangle whose hypotenuse is 3 ft.

(26.) If a is the hypotenuse of an isosceles right triangle, the side is $\frac{1}{2}a\sqrt{2}$.

(27.) Find the side of an isosceles right triangle whose hypotenuse is 32.174 dkm.

(28.) Find the base of an isosceles triangle whose side is 4 ft. and whose vertex angle is 30°.

(29.) If one of the equal sides of an isosceles triangle is R and the vertex angle is 30°, show that the base is $R\sqrt{2-\sqrt{3}}$.



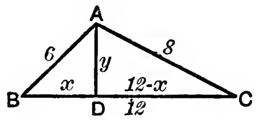
Hint.

$$\left.\begin{aligned} h &= \tfrac{1}{2}R \\ x &= \tfrac{1}{2}R\sqrt{3} \end{aligned}\right\} \qquad \S\ 373(23)$$

$$y = R - x.$$

$$a^2 = h^2 + y^2.$$

(30.) Having given a triangle whose sides are 6, 8, and 12, find its altitude upon the side 12.



Solution.—In the triangle ABD, $y^2+x^2=36$. § 317

In the triangle ADC, $y^2+(12-x)^2=64$.

Combine the two equations and eliminate y.

$$y^2+x^2=36 \quad (1)$$

$$y^2-24x+x^2=-80 \quad (2)$$

$$24x=116$$

$$x=\tfrac{29}{6}=4\tfrac{5}{6}.$$

Substituting this value in (1),

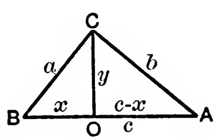
$$y^2+(\tfrac{29}{6})^2=36$$

$$36y^2=455$$

$$6y=\sqrt{455}=21.33+$$

$$y=3.55+$$

(31.) In a triangle whose sides are a, b, and c, find the three altitudes.



Solution.—In the triangle CBO, $x^2+y^2=a^2$. (1) } § 317

In the triangle CAO, $(c-x)^2+y^2=b^2$. (2) }

Simplifying and combining,

$$x^2+y^2=a^2$$
$$x^2-2cx+y^2=b^2-c^2$$
$$2cx=a^2-b^2+c^2$$
$$x=\frac{a^2+c^2-b^2}{2c}$$

Substituting value of x in (1),

$$\left(\frac{a^2+c^2-b^2}{2c}\right)^2+y^2=a^2$$
$$y^2=a^2-\left(\frac{a^2+c^2-b^2}{2c}\right)^2$$
$$y=\sqrt{a^2-\left(\frac{a^2+c^2-b^2}{2c}\right)^2}.$$

This result may be factored and arranged for logarithmic computation as follows:

$$y=\sqrt{a^2-\left(\frac{a^2+c^2-b^2}{2c}\right)^2}=\sqrt{\left(a+\frac{a^2+c^2-b^2}{2c}\right)\left(a-\frac{a^2+c^2-b^2}{2c}\right)}$$
$$=\sqrt{\left(\frac{2ac+a^2+c^2-b^2}{2c}\right)\left(\frac{2ac-a^2-c^2+b^2}{2c}\right)}$$
$$=\sqrt{\frac{1}{c^2}\left(\frac{(a+c)^2-b^2}{2}\right)\left(\frac{b^2-(a-c)^2}{2}\right)}.$$

Multiplying each fraction by $\frac{2}{2}$, and factoring,

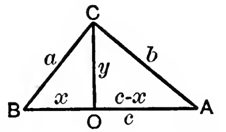
$$y=\sqrt{\frac{4}{c^2}\left(\frac{a+b+c}{2}\right)\left(\frac{a+c-b}{2}\right)\left(\frac{a+b-c}{2}\right)\left(\frac{b+c-a}{2}\right)}.$$

Let $$\frac{a+b+c}{2}=s.$$

Then $$\frac{a+b+c}{2}-b=s-b. \qquad \text{Ax. 3}$$

Whence $$\frac{a+c-b}{2}=s-b.$$

In same manner $\frac{a+b-c}{2}=s-c$, and $\frac{b+c-a}{2}=s-a$.



Substituting these values under radical and extracting root,

$$y=\frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}.$$

The other altitudes are

$$\frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$$

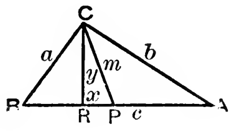
and $$\frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}.$$

(32.) Having given the sides of a triangle equal to 375.49, 289.63, and 231.19, find its three altitudes.

(33.) If the sides of a triangle are 27.931 m., 2175.4 cm., and 296.53 dcm., what are the lengths in feet of (1) the altitude upon the greatest side, and (2) the segments into which it divides that side?

Hint.—After finding the altitude, the segments can easily be found by logarithms, since (§ 318) $x=\sqrt{a^2-y^2}=\sqrt{(a-y)(a+y)}$.

(34.) Compute the medians of a triangle whose sides are a, b, and c.



Solution.—In the triangle CRP, $m^2=x^2+y^2$. (1)

In the triangle CRA, $y^2+\left(\frac{c}{2}+x\right)^2=b^2$. (2)

In the triangle CBR, $y^2+\left(\frac{c}{2}-x\right)^2=a^2$. (3)

§ 317

Simplifying, $y^2+\frac{c^2}{4}+cx+x^2=b^2.$ (2)

$y^2+\frac{c^2}{4}-cx+x^2=a^2.$ (3)

Adding, $2y^2+\frac{c^2}{2}+2x^2=a^2+b^2.$

Transposing, $2(x^2+y^2)=a^2+b^2-\frac{c^2}{2}=\frac{2(a^2+b^2)-c^2}{2}$

$$x^2+y^2=\frac{2(a^2+b^2)-c^2}{4}.$$

But $x^2+y^2=m^2.$ (1)

Therefore $m^2=\frac{2(a^2+b^2)-c^2}{4}.$

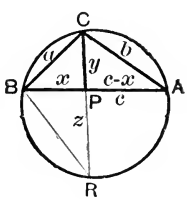
$$m=\tfrac{1}{2}\sqrt{2(a^2+b^2)-c^2}.$$

The other medians are $\frac{1}{2}\sqrt{2(b^2+c^2)-a^2}$ and $\frac{1}{2}\sqrt{2(c^2+a^2)-b^2}$.

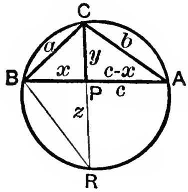
(35.) Having given the three sides of a triangle equal to 3, 5, and 7, find its three medians.

(36.) If two sides and one of the diagonals of a parallelogram are respectively 24, 31, and 28, what is the length of the other diagonal?

(37.) In a triangle whose sides are a, b, and c, compute the bisector of the angle opposite c.



Solution.—Circumscribe a circle about the triangle, produce the bisector to meet the circumference, and draw BR. Then, in the triangles BCR and CPA, the angle R equals the angle O and angle BCR equals the angle PCA. § 201



Therefore $$\frac{y+z}{b}=\frac{a}{y}.$$ § 274

Whence $$y^2+yz=ab. \quad (1)$$ § 250

But $$\frac{a}{b}=\frac{x}{c-x}.$$ § 327

Whence $$bx=ac-ax,$$ § 250

$$x=\frac{ac}{a+b}, \ (2); \text{ and } c-x=\frac{bc}{a+b}. \ (3)$$

But $$(c-x)\times x=y\times z.$$ § 320

Substituting values for x and $(c-x)$ from (2) and (3)

$$yz=\frac{abc^2}{(a+b)^2}. \quad (4)$$

Subtracting (4) from (1) $y^2=ab-\frac{abc^2}{(a+b)^2}=ab\left(1-\frac{c^2}{(a+b)^4}\right).$

$$\boldsymbol{y=\sqrt{ab\left(1-\frac{c^2}{(a+b)^2}\right)}.}$$

This result may be factored and arranged for logarithmic computation as follows:

$$\sqrt{ab\left(1-\frac{c^2}{(a+b)^2}\right)}=\sqrt{ab\left(1+\frac{c}{a+b}\right)\left(1-\frac{c}{a+b}\right)}$$

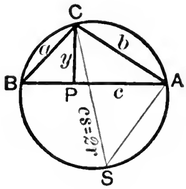
$$=\sqrt{ab\left(\frac{a+b+c}{a+b}\right)\left(\frac{a+b-c}{a+b}\right)}$$

Multiplying both fractions by $\frac{2}{2}$, and extracting root,

$$y=\frac{2}{a+b}\sqrt{ab\left(\frac{a+b+c}{2}\right)\left(\frac{a+b-c}{2}\right)}=\boldsymbol{\frac{2}{a+b}\sqrt{abs(s-c)}}.$$

(38.) If the sides of a triangle are 219.57, 178.35, and 153.94 ft., find the length of the bisector of the angle opposite the greatest side.

(39.) If the sides of a triangle are a, b, and c, find the radius of the circumscribed circle.



Solution.—Suppose the diameter CS of the circle to be drawn from C. Draw SA and the altitude CP.

Then in the right triangles CSA and CBP the angle CAS is equal to the angle P (§ 202), and the angle S is equal to the angle B. § 201

Therefore the triangles are similar, and

$$\frac{2r}{a}=\frac{b}{y}.$$

Hence $$2ry=ab.$$

And $$r=\frac{ab}{2y}.$$

But by Problem (31) $$y=\frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}.$$

Substituting this value,

$$r=\frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}.$$

(40.) If the sides of a triangle are 125.76, 119.53, and 98.991 ft. in length, find the radius of the circumscribing circle expressed in meters.

PLANE GEOMETRY

BOOK IV

AREAS OF POLYGONS

374. *Def.*—The **area** of a surface is the ratio of that surface to another surface taken as the unit.

The unit surface may have any size or shape, but the most common and convenient unit is a square having its side equal to the unit of length, as a square inch, a square mile, etc.

375. *Def.*—**Equivalent** figures are figures having equal areas.

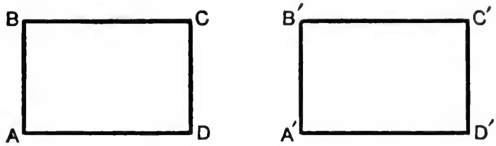
We may observe (1) figures of the same *shape* are *similar*.
(2) figures of the same *size* are *equivalent*.
(3) figures of the same *shape and size* are *equal*.

376. *Defs.*—The **bases** of a parallelogram are the side upon which it is supposed to stand and the opposite side.

The **altitude** is the perpendicular distance between the bases.

PROPOSITION I. THEOREM

377. *Two rectangles having equal bases and equal altitudes are equal.*



GIVEN—two rectangles, AC and $A'C'$, having equal bases, AD and $A'D'$, and equal altitudes, AB and $A'B'$.

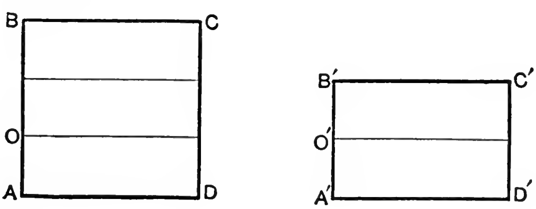
TO PROVE the rectangles equal.

Make AD coincide with its equal $A'D'$.
Then AB will take the direction of $A'B'$. § 18
And B will fall on B'. Hyp.
That is, AB will coincide with $A'B'$.
Similarly DC will coincide with $D'C'$.
And therefore BC will coincide with $B'C'$. Ax. *a*
Hence the rectangles coincide throughout and are equal. § 15

Q. E. D.

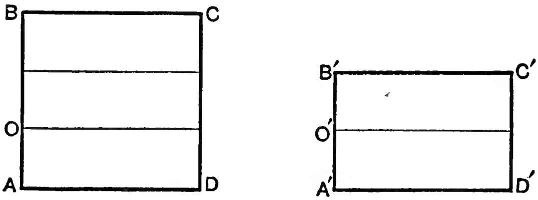
PROPOSITION II. THEOREM

378. *Two rectangles having equal bases are to each other as their altitudes.*



GIVEN—two rectangles AC and $A'C'$, having equal bases, AD and $A'D'$.

TO PROVE $\frac{\text{rect. } AC}{\text{rect. } A'C'} = \frac{AB}{A'B'}$.



CASE I. *When the altitudes, AB and A'B', are commensurable.*

Suppose AO, the common measure of the altitudes, is contained in AB three times and in $A'B'$ twice.

Then $$\frac{AB}{A'B'} = \frac{3}{2}.$$ § 180

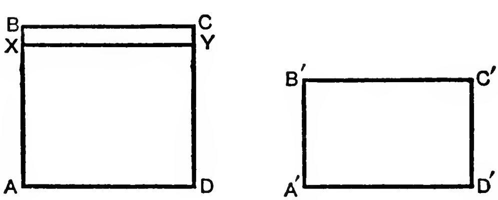
Through the several points of division draw parallels to the bases.

The rectangle AC will be divided into three rectangles and $A'C'$ into two, all five of which will be *equal*. § 377

Hence $$\frac{\text{rect. } AC}{\text{rect. } A'C'} = \frac{3}{2}.$$ § 180

Therefore $$\frac{\text{rect. } AC}{\text{rect. } A'C'} = \frac{AB}{A'B'}.$$ Ax. I

CASE II. *When the altitudes, AB and A'B', are incommensurable.*



Suppose $A'B'$ to be divided into any number of equal parts and apply one of these parts to AB as a measure as often as it will be exactly contained.

Since AB and $A'B'$ are incommensurable, there will be a remainder XB, less than one of these parts.

Draw XY parallel to the base.

Since AX and $A'B'$ are constructed commensurable,

$$\frac{\text{rect. } AY}{\text{rect. } A'C'} = \frac{AX}{A'B'}. \qquad \text{Case I}$$

Now suppose the number of parts into which $A'B'$ is divided to be indefinitely increased.

We can thus make each part as small as we please.

But the remainder XB will always be less than one of these parts.

Therefore we can make XB less than any assigned quantity, though never zero.

That is, AX approaches AB as its limit. § 185

Likewise rect. AY approaches rect. AC as its limit.

Hence $\frac{AX}{A'B'}$ approaches $\frac{AB}{A'B'}$ as its limit.
Also $\frac{\text{rect. } AY}{\text{rect. } A'C'}$, approaches $\frac{\text{rect. } AC}{\text{rect. } A'C'}$ as its limit. } § 190

But since $$\frac{\text{rect. } AY}{\text{rect. } A'C'} = \frac{AX}{A'B'},$$

then $$\frac{\text{rect. } AC}{\text{rect. } A'C'} = \frac{AB}{A'B'}. \qquad \text{§ 186}$$

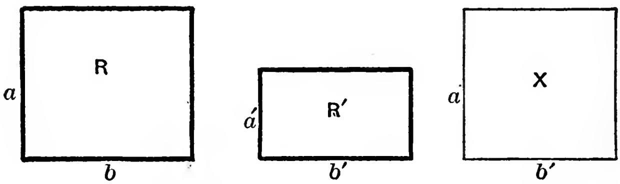
[If two variables are always equal and each approaches a limit, the limits are equal.] Q. E. D.

379. COR. *Two rectangles having equal altitudes are to each other as their bases.*

Hint.—AD and $A'D'$ may be regarded as the altitudes, and AB and $A'B'$ as the bases.

PROPOSITION III. THEOREM

380. *Any two rectangles are to each other as the products of their bases and altitudes.*



GIVEN—any two rectangles, R and R', their bases being b and b', and altitudes a and a'.

TO PROVE $$\frac{R}{R'}=\frac{a\times b}{a'\times b'}.$$

Construct rectangle X, having the same base as R' and altitude as R.

Then $$\frac{R}{X}=\frac{b}{b'}.$$ § 379

[Two rectangles having equal altitudes are to each other as their bases.]

And $$\frac{X}{R'}=\frac{a}{a'}.$$ § 378

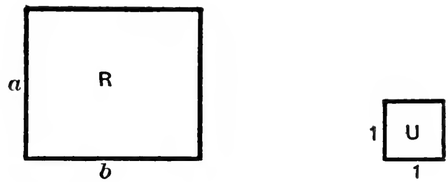
[Two rectangles having equal bases are to each other as their altitudes.]

Multiplying, $$\frac{R}{X}\times\frac{X}{R'}=\frac{b}{b'}\times\frac{a}{a'},$$

or $$\frac{R}{R'}=\frac{a\times b}{a'\times b'}.$$ Q. E. D.

PROPOSITION IV. THEOREM

381. *The area of a rectangle equals the product of its base and altitude, provided the unit of area is a square whose side is the linear unit.*



GIVEN—the rectangle R and a square U with each side a linear unit.

TO PROVE—area of $R = a \times b$, provided U is the unit of area.

$$\frac{R}{U} = \frac{a \times b}{1 \times 1} = a \times b. \qquad \S\ 380$$

[Two rectangles are to each other as the products of their bases by their altitudes.]

But $$\frac{R}{U} = \text{area of } R. \qquad \S\ 374$$

[The area of a surface is the ratio of that surface to the unit surface.]

Therefore area of $R = a \times b$,

provided U is the unit of area. Ax. 1

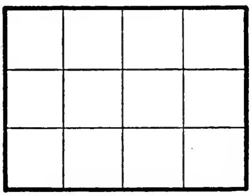
Q. E. D.

382. *Remark.*—Hereafter it is to be understood without any express proviso that we take as the unit of area a square whose side is the linear unit.

383. COR. *The area of any square equals the second power of its side.*

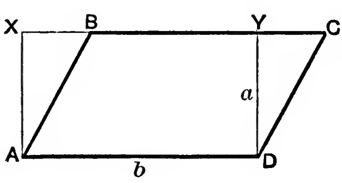
This fact is the origin of the custom of calling the second power of a number its "square."

384. *Remark.*—When the base and altitude of a rectangle each contain the linear unit an exact number of times, Proposition IV. becomes evident to the eye. Thus, if the base contain four and the altitude three linear units, the figure may be divided into twelve unit squares.



PROPOSITION V. THEOREM

385. *The area of a parallelogram equals the product of its base and altitude.*



GIVEN—the parallelogram *ABCD*, with base b and altitude a.

TO PROVE the area of $ABCD = a \times b$.

Draw AX and DY perpendiculars between the parallels AD and BC.

Then $ADYX$ is a rectangle, having the same base and altitude as the parallelogram.

Right triangle AXB = right triangle DYC. (Why?)

Take away the right triangle DYC from the whole figure, and we have left the rectangle $ADYX$.

Take away the right triangle AXB from the whole figure, and we have left the parallelogram $ABCD$.

Therefore area $ADYX$ = area $ABCD$. Ax. 3

But area $ADYX = a \times b$. § 381

[The area of a rectangle equals the product of its base by its altitude.]

Therefore area $ABCD = a \times b$. Ax. 1

Q. E. D.

386. Cor. I. *Parallelograms having equal bases and equal altitudes are equivalent.*

387. Cor. II.—*Any two parallelograms are to each other as the products of their bases and altitudes.*

Hint.—Let the areas of the parallelograms be P and P', their bases b and b', and altitudes a and a'.

Then $P = ab$ and $P' = a'b'$.

And $\frac{P}{P'} = \frac{ab}{a'b'}$.

388. Cor. III. *Two parallelograms having equal bases are to each other as their altitudes.*

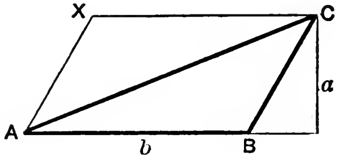
$$\left(\frac{P}{P'} = \frac{a \times b}{a' \times b} = \frac{a}{a'}.\right)$$

389. Cor. IV. *Two parallelograms having equal altitudes are to each other as their bases.*

$$\left(\frac{P}{P'} = \frac{a \times b}{a \times b'} = \frac{b}{b'}.\right)$$

PROPOSITION VI. THEOREM

390. *The area of a triangle equals one-half the product of its base and altitude.*



GIVEN the triangle ABC with base b and altitude a.

TO PROVE area $ABC = \frac{1}{2} a \times b$.

From C draw CX parallel to AB.

From A draw AX parallel to BC.

Then the figure $ABCX$ is a parallelogram. § 114

and the triangle $ABC = \frac{1}{2}$ the parallelogram $ABCX$. § 116

[The diagonal of a parallelogram divides it into two equal triangles.]

But area paral. $ABCX = a \times b$. § 385

[The area of a parallelogram equals the product of its base and altitude.]

Therefore area triangle $ABC = \frac{1}{2} a \times b$. Ax. 8

Q. E. D.

391. COR. I. *Triangles having equal bases and equal altitudes are equivalent.*

392. COR. II. *Any two triangles are to each other as the products of their bases and altitudes.*

$$\left(\frac{P}{P'} = \frac{\frac{1}{2}ab}{\frac{1}{2}a'b'} = \frac{ab}{a'b'}.\right)$$

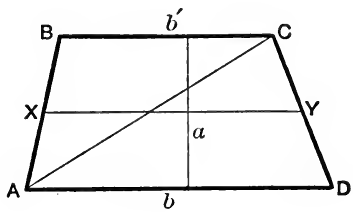
393. COR. III. *Two triangles having equal bases are to each other as their altitudes.*

394. COR. IV. *Two triangles having equal altitudes are to each other as their bases.*

395. *Def.*—The **altitude** of a trapezoid is the perpendicular distance between its bases.

PROPOSITION VII. THEOREM

396. *The area of a trapezoid equals the product of its altitude and one-half the sum of its bases.**



GIVEN—the trapezoid $ABCD$ with altitude a and bases b and b'.

TO PROVE the area of $ABCD = \frac{1}{2}(b+b')a$.

Draw the diagonal AC.

Then area triangle $ADC = \frac{1}{2}ab$, area triangle $ABC = \frac{1}{2}ab'$. § 390

[The area of a triangle equals one-half the product of its base and altitude.]

Adding, area trapezoid $ABCD = \frac{1}{2}ab + \frac{1}{2}ab'$. Ax. 11

$= \frac{1}{2}(b+b')a$. Q. E. D.

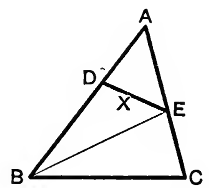
397. COR. *The area of a trapezoid equals the product of its altitude and the line joining the middle points of the non-parallel sides.*

Hint.—Combine § 135 with the above proposition.

* The ancient Egyptians attempted to find the area of a field in the form of a trapezoid, in which $AB = CD$, by multiplying half the sum of its parallel sides by *one of its other sides*, an incorrect method.

PROPOSITION VIII. THEOREM

398. *The areas of two triangles which have an angle of one equal to an angle of the other are to each other as the products of the sides including those angles.*



GIVEN—the triangles ADE and ABC placed so that their equal angles coincide at A.

TO PROVE $$\frac{\text{area } ADE}{\text{area } ABC}=\frac{AD\times AE}{AB\times AC}.$$

Draw BE and denote the triangle ABE by X.

Then, regarding the bases of X and ADE as AB and AD, they will have a common altitude, the perpendicular from E to AB. Likewise X and ABC have bases AE and AC and a common altitude, the perpendicular from B to AC.

Therefore $$\frac{\text{area } ADE}{\text{area } X}=\frac{AD}{AB}$$

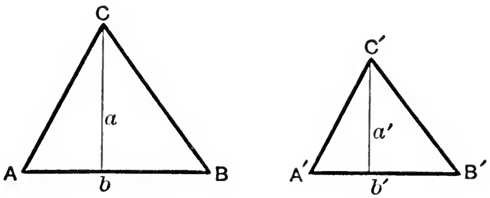
and $$\frac{\text{area } X}{\text{area } ABC}=\frac{AE}{AC}.$$ § 394

[Triangles having equal altitudes are to each other as their bases.]

Multiplying, $$\frac{\text{area } ADE}{\text{area } ABC}=\frac{AD\times AE}{AB\times AC}.$$ Q. E. D.

PROPOSITION IX. THEOREM

399. *The areas of two similar triangles are to each other as the squares of any two homologous sides.*



GIVEN—two similar triangles ABC and $A'B'C'$, b and b' being homologous sides.

TO PROVE $$\frac{\text{area } ABC}{\text{area } A'B'C'} = \frac{b^2}{b'^2}.$$

Draw the altitudes a and a'.

Then $$\frac{\text{area } ABC}{\text{area } A'B'C'} = \frac{a \times b}{a' \times b'} = \frac{a}{a'} \times \frac{b}{b'}. \qquad \S\ 392$$

[Two triangles are to each other as the products of their bases and altitudes.]

But $$\frac{a}{a'} = \frac{b}{b'}. \qquad \S\ 290$$

[Homologous altitudes of similar triangles have the same ratio as homologous sides.]

Substitute, in the previous equation, $\frac{b}{b'}$ for $\frac{a}{a'}$.

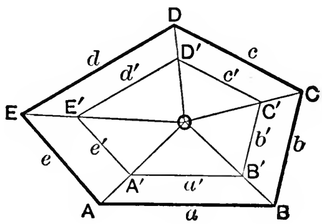
Then $$\frac{\text{area } ABC}{\text{area } A'B'C'} = \frac{b}{b'} \times \frac{b}{b'} = \frac{b^2}{b'^2}.$$ Q. E. D.

SUMMARY: $\frac{\text{area } ABC}{\text{area } A'B'C'} = \frac{a \times b}{a' \times b'} = \frac{a}{a'} \times \frac{b}{b'} = \frac{b}{b'} \times \frac{b}{b'} = \frac{b^2}{b'^2}.$

400. *Exercise.*—Prove the last proposition by means of Proposition VIII.

PROPOSITION X. THEOREM

401. *The areas of two similar polygons are to each other as the squares of any two homologous sides.*



GIVEN—the similar polygons $ABCDE$ and $A'B'C'D'E'$, with sides a, b, c, d, e, and a', b', c', d', e', and areas M and M' respectively.

TO PROVE $$\frac{M}{M'}=\frac{a^2}{a'^2}.$$

If $ABCDE$ and $A'B'C'D'E'$ are radially placed so that O, the centre of similitude, is within the two polygons, the triangles OAB, OBC, OCD, etc., are respectively similar to $OA'B'$, $OB'C'$, $OC'D'$, etc. § 285

Then $\frac{\text{area } OAB}{\text{area } OA'B'}=\frac{a^2}{a'^2}$, $\frac{\text{area } OBC}{\text{area } OB'C'}=\frac{b^2}{b'^2}$, $\frac{\text{area } OCD}{\text{area } OC'D'}=\frac{c^2}{c'^2}$, etc. § 399

[The areas of two similar triangles are to each other as the squares of any two homologous sides.]

But $$\frac{a^2}{a'^2}=\frac{b^2}{b'^2}=\frac{c^2}{c'^2}=\text{etc.}$$ § 274

Hence $$\frac{\text{area } OAB}{\text{area } OA'B'}=\frac{\text{area } OBC}{\text{area } OB'C'}=\frac{\text{area } OCD}{\text{area } OC'D'}=\text{etc.}=\frac{a^2}{a'^2}.$$

Ax. 1

Therefore $\dfrac{\text{area } OAB + \text{area } OBC + \text{area } OCD + \text{etc.}}{\text{area } OA'B' + \text{area } OB'C' + \text{area } OC'D' + \text{etc.}} = \dfrac{a^2}{a'^2}$. § 265

But area OAB + area OBC + area OCD + etc. $= M$, Ax. 11
and area $OA'B'$ + area $OB'C'$ + area $OC'D'$ + etc. $= M'$.

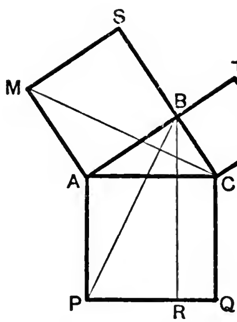
Therefore $\dfrac{M}{M'} = \dfrac{a^2}{a'^2}$.

Q. E. D.

402. Cor. Since $\dfrac{a}{a'}$, = ratio of similitude, *the ratio of the areas of two similar polygons equals the square of their ratio of similitude.*

PROPOSITION XI. THEOREM

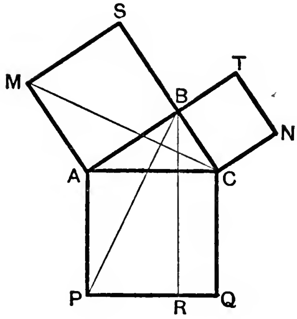
403. *The square described on the hypotenuse of a right triangle is equivalent to the sum of the squares on the other two sides.**



Given—the right triangle ABC and the squares described on its three sides.

To prove—area of square AQ = area of square BN + area of square BM.

* Proposition XI. was discovered by Pythagoras (about 550 B.C.) and is usually known as the Pythagorean theorem. The proof here given is however due to Euclid (about 300 B.C.), that of Pythagoras being unknown.



Now *ABT* and *CBS* are straight lines. § 29

Join *MC* and *BP* and draw *BR* parallel to *AP*.

Triangle* *AMC* = triangle *ABP*. § 79

[Having two sides and the included angle equal, viz., *AM* = *AB*, being sides of a square; likewise *AC* = *AP*, and angle *CAM* = angle *PAB*, since each consists of a right angle and the common angle *BAC*.]

But rectangle *AR* ⌯ twice triangle *ABP*. §§ 381, 390

[Having the same base *AP* and the same altitude, the distance between the parallels *AP* and *BR*.]

Likewise square *BM* ⌯ twice triangle *AMC*.

Therefore rectangle *AR* ⌯ square *BM*. Ax. 7

Likewise we may prove

rectangle *CR* ⌯ square *BN*.

Adding,

rect. *AR* + rect. *CR* ⌯ sq. *BM* + sq. *BN*.

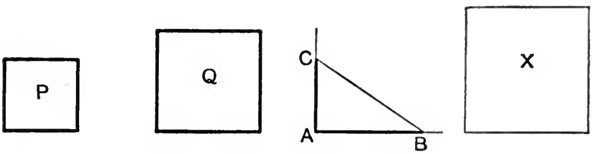
Or sq. *AQ* ⌯ sq. *BM* + sq. *BN*. Q. E. D.

404. Cor. *The square on either side about the right angle is equivalent to the difference of the squares on the hypotenuse and on the other side.*

* The eye will interpret this equality by conceiving the triangle *AMC* to turn around *A* as a pivot until *AM* falls on *AB*.

405. *Remark.*—Proposition XV., Book III., differs from the preceding proposition in that the squares of the sides in the former referred to the *algebraic* squares, that is, the second power of the numbers representing the sides, whereas in the latter case the squares are *geometric.* Inasmuch as the algebraic square measures the geometric square (§ 383), the truth of either of the two propositions involves the truth of the other.

406. CONSTRUCTION. *To construct a square equivalent to the sum of two given squares.*



GIVEN two squares P and Q.

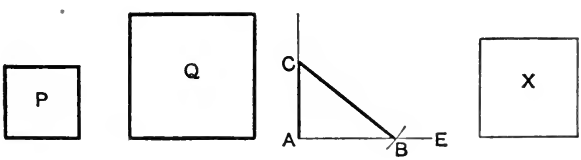
TO CONSTRUCT a square equivalent to $P + Q$.

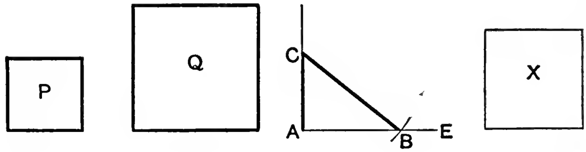
Construct a right angle A and on its sides lay off AB and AC equal respectively to the sides of Q and P. Join BC.

Construct the square X having its side equal to BC.

X is the required square. (Why?) Q. E. F.

407. CONSTRUCTION. *To construct a square equivalent to the difference of two given squares.*





GIVEN two squares, P and Q, of which P is the smaller.

TO CONSTRUCT a square equivalent to $Q - P$.

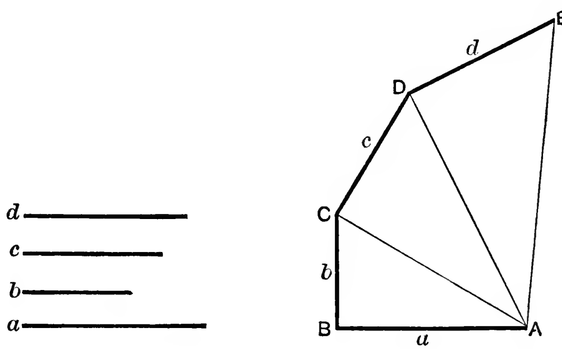
Construct a right angle A, and on one side lay off AC equal to the side of P.

Then from C as a centre, with the side of Q as a radius, describe an arc cutting AE at B.

Construct the square X having its side equal to AB.

X is the required square. (Why?) Q. E. F.

408. CONSTRUCTION. *To construct a square equivalent to the sum of any number of given squares.*



GIVEN a, b, c, d, the sides of given squares.

TO CONSTRUCT — a square equivalent to the sum of these given squares.

Draw AB equal to a.

At B draw BC perpendicular to AB and equal to b; join AC.

At C draw CD perpendicular to AC and equal to c; join AD.

At D draw DE perpendicular to AD and equal to d; join AE.

The square constructed on AE as a side is the square required.

Proof.—

Sq. on AE ⌯ sq. on d + sq. on AD.

⌯ sq. on d + sq. on c + sq. on AC.

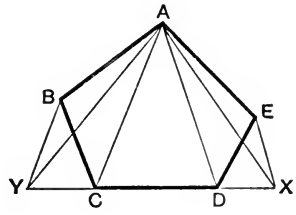
⌯ sq. on d + sq. on c + sq. on b + sq. on a.

Q. E. F.

409. *Remark.*—The foregoing construction enables a draughtsman to construct a line whose length is equal to any square root.

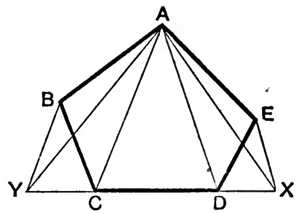
Thus suppose we wish to construct a line equal to $\sqrt{3}$ inches. Lay off a, b, c, one inch each; then $AD = \sqrt{3}$ inches.

410. CONSTRUCTION. *To construct a triangle equivalent to a given polygon.*



GIVEN the polygon $ABCDE$.

TO CONSTRUCT a triangle equivalent to it.



Join any two alternate vertices as A and D.

Draw EX parallel to AD and meeting CD produced at X. Join AX.

The polygon $ABCX$ has one less side than the original polygon, but is equivalent to it.

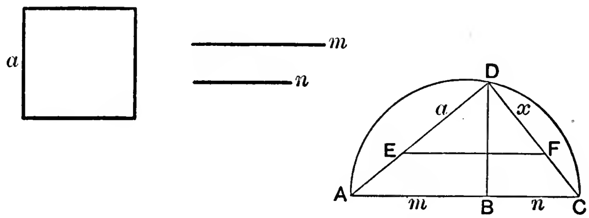
For the part $ABCD$ is common,

and triangle ADE ⇔ triangle ADX. § 391

[Having the same base AD and the same altitude, the distance between the parallels AD and EX.]

In like manner reduce the number of sides of the new polygon $ABCX$, and thus continue until the required triangle AXY is obtained. Q. E. F.

411. CONSTRUCTION. *To construct a square which shall have a given ratio to a given square.*



GIVEN—a the side of a given square and $\frac{n}{m}$ the given ratio.

TO CONSTRUCT—a square which shall have the ratio $\frac{n}{m}$ to the given square.

Draw the straight line AB equal to m and produce it making BC equal to n.

Upon AC as a diameter construct a semicircle.

Erect the perpendicular BD meeting the circumference at D, and join DA and DC.

On DA lay off DE equal to a and draw EF parallel to AC.

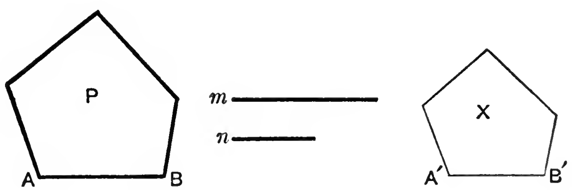
Then DF, or x, is the side of the square required.

Proof: $\dfrac{\text{square on } x}{\text{square on } a} = \dfrac{x^2}{a^2}$ § 383

$= \left(\dfrac{x}{a}\right)^2 = \left(\dfrac{DC}{DA}\right)^2 = \dfrac{\overline{DC}^2}{\overline{DA}^2} = \dfrac{BC}{AB}.$ §§ 272, 313

$= \dfrac{n}{m}.$ Q. E. D.

412. CONSTRUCTION. *To construct a polygon similar to a given polygon and having a given ratio to it.*



GIVEN the polygon P, and the ratio $\dfrac{n}{m}$.

TO CONSTRUCT—a polygon similar to P, and which shall be to P as n is to m.

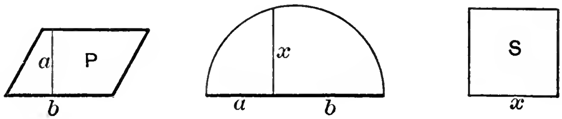
Find a line $A'B'$ such that the square upon it shall be to the square upon AB as n is to m. § 411

Upon $A'B'$, as the homologous side to AB, construct the required similar polygon X. § 302

Proof: $\dfrac{X}{P} = \dfrac{\overline{A'B'}^2}{\overline{AB}^2} = \dfrac{n}{m}.$ (Why?)

Q. E. D.

413. CONSTRUCTION. *To construct a square equivalent to a given parallelogram.*



GIVEN a parallelogram P with base b and altitude a.

TO CONSTRUCT a square equivalent to P.

Construct x a mean proportional between a and b. § 316
Upon x construct the required square S.

Proof.—By construction $\frac{a}{x}=\frac{x}{b}$.

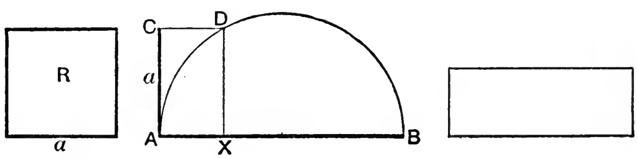
Hence $x^2=a\times b$. § 250
That is, area S=area P. §§ 383, 385

Q. E. D.

414. *Exercise.*—Show that a square can be constructed equivalent to a given triangle by taking for its side a mean proportional between the altitude and half the base.

415. *Exercise.*—Show that a square can be constructed equivalent to a given polygon by first reducing the polygon to an equivalent triangle and then constructing a square equivalent to the triangle.

416. CONSTRUCTION. *To construct a rectangle equivalent to a given square, and having the sum of its base and altitude equal to a given line.*



GIVEN—a, the side of the given square R, and AB, the given line.

TO CONSTRUCT—a rectangle equivalent to R and having its base and altitude together equal to AB.

Upon AB as a diameter construct a semicircle.

Draw CD parallel to AB and at a distance from it equal to a.

From D the intersection of CD with the circumference draw DX perpendicular to AB.

The rectangle having AX for its altitude and XB for its base is the required rectangle.

Proof: $$\frac{AX}{DX}=\frac{DX}{XB}.$$ § 315

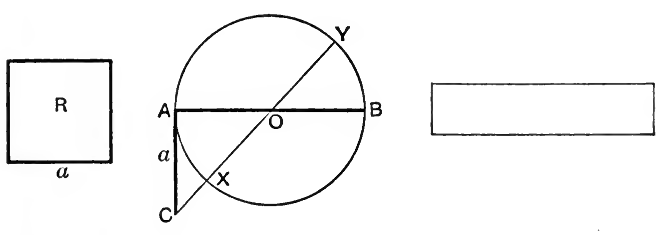
Hence $AX \times XB = \overline{DX}^2$. § 250

That is, area rectangle = area square. §§ 381, 383

Also $AX + XB = AB$. Q. E. F.

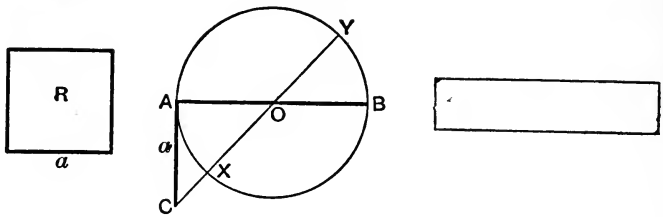
417. *Remark.*—§ 416 may be stated: To find two straight lines of which the sum and product are given.

418. CONSTRUCTION. *To construct a rectangle equivalent to a given square, and having the difference of its base and altitude equal to a given line.*



GIVEN a, the side of the square R, and the line AB.

TO CONSTRUCT—a rectangle equivalent to R, and having the difference of its base and altitude equal to AB.



Upon AB as a diameter construct a circumference.

At A draw the tangent AC equal to a.

Draw CXY through the centre meeting the circumference in X and Y.

Then the rectangle having its base equal to CY and its altitude equal to CX is the required rectangle.

Proof: $$\frac{CX}{a}=\frac{a}{CY}.$$ § 321

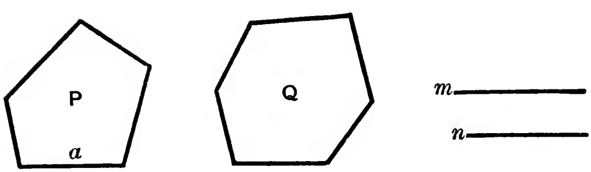
Whence $$CX \times CY = a^2.$$ § 250

Or, area rectangle = area square. §§ 381, 383

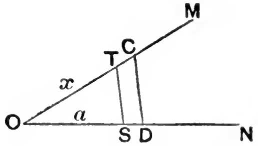
Also XY, the difference between CY and CX, is a diameter of the circle, and therefore equal to AB. Q. E. F.

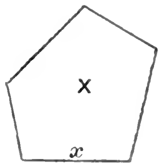
419. *Remark.*—§ 418 may be stated: To find two straight lines of which the difference and product are given.

420. CONSTRUCTION. *To construct a polygon similar to a given polygon and equivalent to another given polygon.**



* Pythagoras (about 550 B.C.) first solved this problem.





GIVEN the polygons P and Q.

TO CONSTRUCT—a polygon similar to P and equivalent to Q.

Construct squares equivalent to P and Q. § 415

Let n and m be the sides of these squares.

From any point O draw two lines OM and ON, and on these lay off OC equal to m and OD equal to n. On OD lay off OS equal to a, a side of P.

Draw parallels giving the fourth proportional OT. § 282

Upon OT, or x, as a side homologous to a, construct a polygon X similar to P. It will also be equivalent to Q.

Proof: $$\frac{X}{P}=\frac{x^2}{a^2}=\frac{m^2}{n^2}=\frac{\text{sq. on } m}{\text{sq. on } n}=\frac{Q}{P}.$$ (Why?)

Therefore X is equivalent to Q and is similar to P by construction. Q. E. F.

PROBLEMS OF DEMONSTRATION

421. The square on the base of an isosceles triangle, whose vertical angle is a right angle, is equivalent to four times the triangle.

422. A quadrilateral is divided into two equivalent triangles by one of its diagonals, if the other diagonal is bisected by the first.

423. The four triangles formed by drawing the diagonals of a parallelogram are all equivalent.

424. If from the middle point of one of the diagonals of a quadrilateral straight lines are drawn to the opposite vertices, these two lines divide the figure into two equivalent parts.

425. If the sides of any quadrilateral are bisected and the points of bisection successively joined, the included figure will be a parallelogram equal in area to half the original figure.

426. A trapezoid is divided into two equivalent parts by the straight line joining the middle points of its parallel sides.

427. The triangle formed by joining the middle point of one of the non-parallel sides of a trapezoid to the extremities of the opposite side is equivalent to one-half the trapezoid.

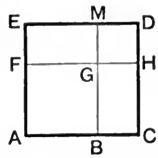
428. If the three sides of a right triangle are the homologous sides of similar polygons described upon them, then the polygon described upon the hypotenuse is equivalent to the sum of the polygons described upon the other two sides.

429. If M is the intersection of the medians of a triangle ABC, the triangle AMB is one-third of ABC.

430. If from the middle point of the base of a triangle lines parallel to the sides are drawn, the parallelogram thus formed is equivalent to one-half the triangle.

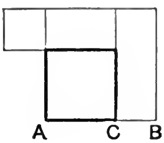
431. Any straight line drawn through the intersection of the diagonals of a parallelogram divides the parallelogram into two equivalent parts.

432. The square described upon the sum of two straight lines is equivalent to the sum of the squares described upon the two lines plus twice their rectangle.



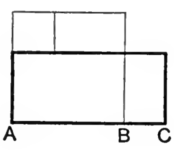
Hint.—Let AB and BC be the given lines.

433. The square described upon the difference of two straight lines is equivalent to the sum of the squares described upon the two lines minus twice their rectangle.



Hint.—Let AB and BC be the given lines.

434. The rectangle whose sides are the sum and the difference of two straight lines is equivalent to the difference of the squares described upon the two lines.



Hint.—Let AB and BC be the given lines.

Question.—To what three formulas of algebra* do the last three problems correspond?

* Euclid gave the geometric proofs of §§ 432–4; but though he may have translated them into algebra, he was probably not acquainted with the algebraic proof. To-day we find it easier to obtain the algebraic formulas first, and then give them the geometric interpretation. This is true in a multitude of cases where the opposite was true among the Greeks.

PROBLEMS OF CONSTRUCTION

435. To divide a triangle into three equivalent triangles by straight lines from one of the vertices to the side opposite.

436. To construct an isosceles triangle equivalent to any given triangle, and having the same base.

437. On a given side, to construct a triangle equivalent to any given triangle.

438. Having given an angle and one of the including sides, to construct a triangle equivalent to a given triangle.

439. To construct a right triangle equivalent to a given triangle.

440. To construct a right triangle equivalent to a given triangle, and having its base equal to a given line.

441. On a given hypotenuse to construct a right triangle equivalent to a given triangle. When is the problem impossible?

442. To draw a straight line through the vertex of a given triangle so as to divide it into two parts having the ratio 2 to 5.

443. To bisect a triangle by a straight line drawn from a given point in one of its sides. § 398

444. On a given side to construct a rectangle equivalent to a given square.

445. To construct a square equivalent to a given triangle.

446. To construct a square equivalent to the sum of two given triangles.

447. On a given side to construct a rectangle equivalent to the sum of two given squares.

448. To construct a square which shall have a given ratio to a given hexagon.

449. Through a given point within any parallelogram to draw a straight line dividing it into two equivalent parts.

PROBLEMS FOR COMPUTATION

450. (1.) Find the area of a parallelogram one of whose sides is 37.53 m., if the perpendicular distance between it and the opposite side is 2.95 dkm.

(2.) Required the area of a rhombus if its diagonals are in the ratio of 4 to 7, and their sum is 16.

(3.) In a right triangle the perpendicular from the vertex of the right angle to the hypotenuse divides the hypotenuse into the segments m and n. Find the area of the triangle.

(4.) If the hypotenuse of an isosceles right triangle is 30 ft., find the number of ares in its area.

(5.) Find the area of an isosceles right triangle if the hypotenuse is equal to a.

(6.) If one of the equal sides of an isosceles triangle is 17 dkm. in length and its base is 30 m., find the area of the triangle.

(7.) Find the area of an isosceles triangle if one of the equal sides is a and its base is b.

(8.) If in the above example $a = 17.163$ hm. and $b = 27.395$ hm., how many acres are there in the triangle?

(9.) Find the area of an equilateral triangle if one of the sides equals 16 m.

(10.) If the side of an equilateral triangle is a, find its area.

(11.) If each side of a triangular park measures 196.37 rds., how many hectares does it contain?

(12.) If the perimeter of an equilateral triangle is 523.65 ft., find its area.

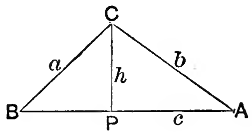
(13.) Find the area of a triangle, if two of its sides are 6 in. and 7 in. and the included angle is 30°.

(14.) Show that, if a and b are the sides of a triangle, the area is $\frac{1}{4}ab$, when the included angle is 30° or 150°; $\frac{1}{4}ab\sqrt{2}$, when the included angle is 45° or 135°; $\frac{1}{4}ab\sqrt{3}$, when the included angle is 60° or 120°.

(15.) Find the area of a triangle, if two of its sides are 43.746 mm. and 15.691 mm., and the included angle is 120°.

(16.) How many square feet are there in the entire surface of a house 50 ft. long, 40 ft. wide, 30 ft. high at the corners, and 40 ft. high at the ridge-pole?

(17.) Find the area of a triangle whose sides are a, b, and c.



Solution.—The area of the triangle $ABC = \frac{c}{2} \times h$.

But $$h = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}.$$ § 373(31)

Whence $$\text{area} = \frac{c}{2} \times \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}$$
$$= \sqrt{s(s-a)(s-b)(s-c)}.$$

(18.) Find the area of a triangle whose sides are 119.3 m., 147.35 m., and 7 dkm.

(19.) Required the area of the quadrilateral $ABCD$, if the four sides AB, BC, CD, and DA measure respectively 63.57, 113.29, 39.637, and 156 ft., and the diagonal $AC = 150.26$ ft.

(20.) If the bases of a trapezoid are respectively 97 m. and 133 m., and its area is 46 ares, find its altitude.

(21.) Find the area of a trapezoid of which the bases are 73 ft. and 57 ft., and each of the other sides is 17 ft.

(22.) Find the area of a trapezoid of which the bases are a and b and the other sides are each equal to d.

(23.) If in the triangle ABC a line MN is drawn parallel to the side AC so that the smaller triangle which it cuts off equals one-third of the whole triangle, find MN in terms of AC.

(24.) Through a triangular field a path runs from one corner to a point in the opposite side 204 yds. from one end, and 357 yds. from the other. What is the ratio of the two parts into which the field is divided?

(25.) If a square and a rhombus have equal perimeters, and the altitude of the rhombus is four-fifths its side, compare the areas of the two figures.

(26.) The altitude upon the hypotenuse of an isosceles right triangle is 3.1572 m. Find the side of an equivalent square.

(27.) If the areas of two triangles of equal altitude are 9 hectares and 324 ares respectively, what is the ratio of their bases?

(28.) A triangle and a rectangle are equivalent. (*a.*) If their bases are equal find the ratio of their altitudes. (*b.*) Compare their bases if their altitudes are equal.

(29.) Two homologous sides of two similar polygons are respectively 12 m. and 36 m. in length, and the area of the first is 180 sq. m. What is the area of the second?

(30.) Two similar fields together contain 579 hectares. What is the area of each if their homologous sides are in the ratio of 7 to 12?

(31.) In a triangle having its base equal to 24 in. and an area of 216 sq. in., a line is drawn parallel to the base through a point 6 in. from the opposite vertex. Find the area of the smaller triangle thus formed.

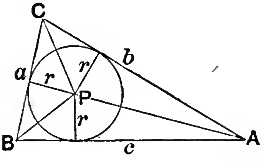
(32.) The altitude of a triangle is a and its base is b; the altitude, homologous to a, of another triangle, similar to the first, is c. Find the altitude, base, and area of a triangle similar to the given triangles and equivalent to their sum.

(33.) Construct a square equivalent to the sum of the squares whose sides are 20, 16, 9, and 5 cm.

(34.) If the sides of a triangle are 113.61 cm., 97.329 cm., and 82.52 cm., find the areas of the parts into which it is divided by the bisector of the angle opposite the first side.

(35.) If to the base b of a triangle the line d is added, how much must be taken from its altitude h that its area may remain unchanged?

(36.) If the sides of a triangle are a, b, and c, find the radius of the inscribed circle.



Solution.—The area of the triangle $CBP = \frac{a}{2} \times r$.

The area of the triangle $CAP = \frac{b}{2} \times r$.

The area of the triangle $BAP = \frac{c}{2} \times r$.

The sum of these areas, or the area of the triangle ABC,

$$=\frac{a+b+c}{2}\times r=sr.$$

But by (17) the area of

$$ABC=\sqrt{s(s-a)(s-b)(s-c)}.$$

Therefore $$sr=\sqrt{s(s-a)(s-b)(s-c)}$$

$$r=\frac{1}{s}\sqrt{s(s-a)(s-b)(s-c)}$$

$$=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

(37.) If the sides of a triangle are 173.52 cm., 125.3 cm., and 96.357 cm., find the radius of the inscribed circle.

PLANE GEOMETRY

BOOK V

REGULAR POLYGONS AND CIRCLES. SYMMETRY WITH RESPECT TO A POINT

451. *Defs.*—A figure turns **half-way round** a point, if a straight line of the figure passing through the point turns through 180°, i. e., half of 360°.

A figure turns **one-third-way round** a point, if a straight line of the figure passing through the point turns through 120°, i. e., one-third of 360°.

In general, a figure turns **one-nth way round** a point if a straight line of the figure passing through the point turns through one-n^{th} of 360°.

452. *Exercise.*—If a figure is turned half-way round on a point as a pivot, i. e., so that *one* straight line of the figure passing through that point turns through 180°, prove that *every other* straight line of the figure passing through that point turns through 180°.

453. *Exercise.*—In the same case, prove that every straight line not passing through the pivot makes after the rotation an angle of 180° with its original position.

454. *Exercise.*—If a figure turns one-third way round, prove that every straight line, whether passing through the pivot or not, makes after the rotation an angle of 120° with its original position.

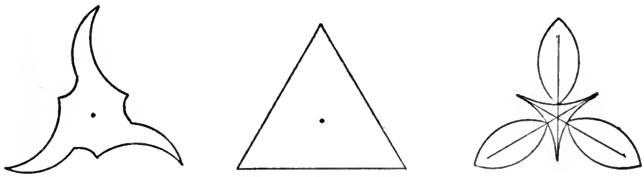
455. *Exercise.*—If a figure turns one-n^{th} way round, prove that every straight line of the figure makes after the rotation an angle equal to $\frac{1}{n}$ of 360° with its original position.

456. *Remark.*—Hence we see the propriety of saying that when one straight line of the figure turns through an angle, the whole figure turns through the same angle.

457. *Defs.*—A figure was defined to be symmetrical with respect to a point, called the **centre of symmetry** (§ 40), if, on being turned *half-way round* on that point as a pivot, the figure coincides with its original position or impression.

To distinguish this kind of symmetry from those which follow, it may be called **two-fold symmetry** with respect to a point.

458. *Def.*—A figure has **three-fold** symmetry with respect to a point, if, on being turned *one-third* way round on that point as a pivot, it coincides with its original impression.

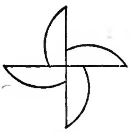


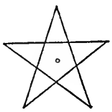
FIGURES POSSESSING THREE-FOLD SYMMETRY WITH RESPECT TO A POINT

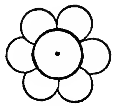
A figure which coincides with its original when turned one-third way round must also coincide when turned *two*-thirds. For, since it *coincides* after the first third, it may then be regarded as the original figure, and will therefore coincide when turned one-third again. When turned the third third the figure has completed one revolution, and each part is in its original position. It is easy to copy one of the above figures on tracing-paper or card-board, cut it out, fit it again to the page, stick a pin through its centre, and turn the figure one-third way round. In Propositions I. and II. it is convenient to think of the original diagram as fixed on the page, while another diagram, as the card-board, revolves upon it.

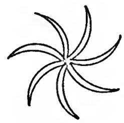
459. *Defs.*—We may define likewise four-fold, five-fold, etc., symmetry. In general a figure has **n-fold symmetry** with respect to a point, called the **centre of symmetry,** if, on being turned about that point one-n^{th} of a revolution, it coincides with its original impression.

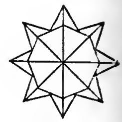
Such a figure will also coincide if turned an n^{th} of a revolution a second, third, fourth time, etc. For after the first n^{th} it becomes the *original figure*, and will therefore coincide when turned one-n^{th} again.


4-FOLD SYMMETRY


5-FOLD SYMMETRY *


6-FOLD SYMMETRY

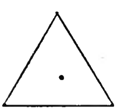

7-FOLD SYMMETRY

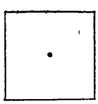

8-FOLD SYMMETRY

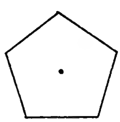
460. *Defs.*—A triangle is **regular,** if it has three-fold symmetry with respect to a point. The point is called the **centre of the triangle.**

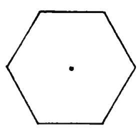
A quadrilateral is **regular,** if it has four-fold symmetry; a pentagon if it has five-fold symmetry, etc.

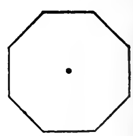
In general a polygon of n sides is **regular,** if it has n-fold symmetry. The centre of symmetry is called the **centre of the polygon.**


REGULAR TRIANGLE


REGULAR QUADRILATERAL


REGULAR PENTAGON


REGULAR HEXAGON


REGULAR OCTAGON

* This figure was used as a badge by the secret society founded by Pythagoras about 550 B.C. for the pursuit of Mathematics and Philosophy. It was supposed to possess mysterious properties, and was called "Health."

PROPOSITION I. THEOREM

461. *Given a regular polygon :*

I. *All its sides are equal.*

II. *All its angles are equal.*

III. *A circle may be circumscribed about it, its centre being the centre of the polygon.*

IV. *A circle may be inscribed in it, its centre being the centre of the polygon.*

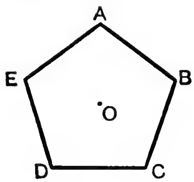


FIG. 1

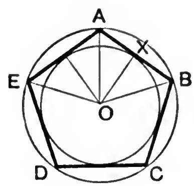


FIG. 2

GIVEN—*ABCDE*, a regular polygon of n sides with centre O.

TO PROVE—I. Its sides are equal.

II. Its angles are equal.

III. A circle can be circumscribed, with centre O.

IV. A circle can be inscribed, with centre O.

I. (Fig. 1.) By definition, the polygon will, after being turned about O one-n^{th} of a revolution, coincide with its original impression. § 460

Any side as AB must therefore take the position previously occupied by some other side.

Since each turn is one-n^{th} of a revolution, n turns are necessary before AB resumes its original position.

Hence in a complete revolution AB must coincide in succession with the n different sides of the polygon.

Hence AB is equal to each of the other sides, and they are all equal to each other. Q. E. D.

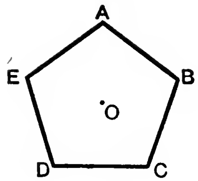


FIG. 1

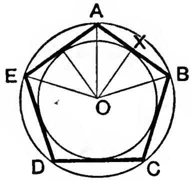


FIG. 2

II. (Fig. 1.) Likewise any angle, as A, must in the n turns necessary for a complete revolution coincide in succession with the n different angles of the polygon.

Hence the angles are all equal. Q. E. D.

III. (Fig. 2.) Since the vertex A always remains at the same distance from O, it describes a circumference whose centre is O.

But it has been shown that the point A coincides successively with B, C, D, etc.

Hence the circumference described by A passes through B, C, D, etc.

That is, this circumference is circumscribed about the polygon and has for its centre the point O. § 218

Q. E. D.

IV. (Fig. 2.) Consider a perpendicular from O upon any side, as OX upon AB.

As the figure revolves, AB coincides successively with each of the other sides, and therefore OX becomes successively perpendicular to each side.

Hence the circumference generated by X, whose radius is OX, passes through the feet of all the perpendiculars from O to the sides.

The sides are therefore all tangent to this circle. § 173

That is, the circle is inscribed in the polygon, and has its centre at O. § 214

Q. E. D.

462. COR. I. *A regular triangle is an equilateral and equiangular triangle. A regular quadrilateral is a square.*

463. COR. II. *Each angle of a regular polygon is* $\frac{2n-4}{n}$ *right angles (n being the number of sides).*

Hint.—By § 66 the sum of all the angles is $2n-4$ right angles.

464. *Def.*—The **radius** of a regular polygon is the radius of the circumscribed circle, that is, the line from the centre to a vertex.

465. *Def.*—The **apothem** of a regular polygon is the radius of the inscribed circle, that is, the perpendicular from the centre to a side.

466. COR. III. *The angles at the centre of a regular polygon between successive radii are all equal, and each is one-n^{th} of four right angles.*

467. *Def.*—Any one of these angles is usually spoken of simply as the **angle at the centre.**

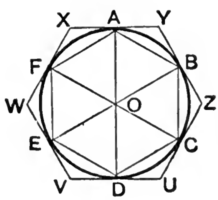
468. COR. IV. *The angle at the centre of a regular polygon is bisected by the apothem.*

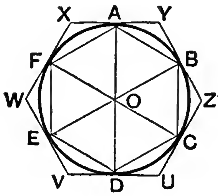
PROPOSITION II. THEOREM

469. *If the circumference of a circle be subdivided into three or more equal arcs:*

I. *Their chords form a regular inscribed polygon, whose centre is the centre of the circle.*

II. *The tangents at the points of division form a regular circumscribed polygon, whose centre is the centre of the circle.*





GIVEN—a circle whose centre is *O* and whose circumference is divided into *n* equal arcs at the points *A, B, C, D,* etc.

TO PROVE—I. The *n* chords *AB, BC,* etc., form a regular polygon, with centre *O.*

II. The *n* tangents *XAY, YBZ,* etc., form a regular polygon, with centre *O.*

I. Revolve the figure one-n^{th} of 360°.

As the figure is turned, the circumference slides along itself. § 159

Since the arcs are each equal to one-n^{th} of the circumference, when *A* reaches *B*, *B* will reach *C*, *C* will reach *D*, etc.

That is, each vertex of the revolved polygon coincides with a vertex of the original polygon.

Since the vertices coincide, the sides which connect them must also coincide. Ax. *a*

Hence the whole polygon coincides with its original impression, and is therefore regular. § 460

Q. E. D.

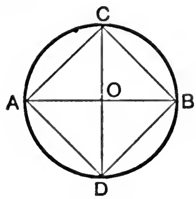
II. We have just proved that when the figure is revolved one-n^{th}, the vertices *A*, *B*, *C*, etc., will coincide respectively with *B*, *C*, *D*, etc., and we know that the circumference will coincide with itself. § 159

Hence the tangents at *A*, *B*, *C*, etc., will coincide respectively with the tangents at *B*, *C*, *D*, etc. §§ 173, 18

Hence the whole circumscribed polygon will coincide with its original impression, and is therefore regular. § 460

Q. E. D.

470. CONSTRUCTION. *To inscribe a regular quadrilateral, or square, in a given circle.*



GIVEN a circle with centre O.

TO CONSTRUCT an inscribed square.

Draw two perpendicular diameters AB and CD.

Join their extremities.

$ACBD$ is the required square.

Proof.—The arcs AC, CB, BD, DA are equal. § 162

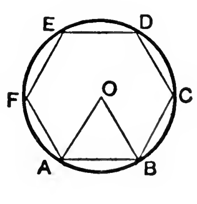
[Subtending equal angles at the centre.]

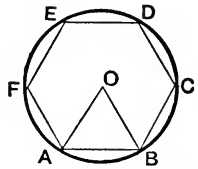
Hence $ACBD$ is a regular quadrilateral. § 469 I

Q. E. D.

471. *Remark.*—A regular polygon of eight sides can be inscribed by bisecting the arcs AC, CB, etc.; and, by continuing the process, regular polygons of sixteen, thirty-two, sixty-four, one hundred and twenty-eight, etc., sides can be inscribed.

472. CONSTRUCTION. *To inscribe a regular hexagon in a given circle.*





GIVEN a circle with centre O.

TO CONSTRUCT a regular inscribed hexagon.

Draw any radius OA.

With A as a centre and a radius equal to OA describe an arc intersecting the circumference at B.

AB is a side of the required regular inscribed hexagon.

Proof.—Join OB.

The triangle OAB is equilateral. Cons.

Hence angle O is 60°, i. e., one-sixth of 360°. § 74

Hence arc AB is one-sixth of the circumference. § 191

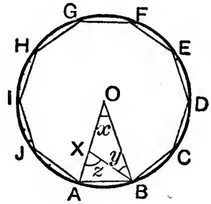
Therefore chord AB is a side of a regular inscribed hexagon. § 469 I

Q. E. D.

473. *Exercise.*—Show that a regular inscribed triangle is formed by joining the alternate vertices A, C, and E.

474. *Remark.*—A regular inscribed polygon of twelve sides can be formed by bisecting the arcs AB, BC, etc.; and, by continuing the process, regular polygons of twenty-four, forty-eight, ninety-six, etc., sides can be inscribed.

475. CONSTRUCTION. *To inscribe a regular decagon in a given circle.*



GIVEN a circle with centre O.

TO CONSTRUCT a regular inscribed decagon.

Divide a radius OA internally in extreme and mean ratio,

i. e., so that $\frac{OA}{OX}=\frac{OX}{XA}$. § 335

With A as a centre and OX as a radius, describe an arc cutting the circumference at B.

AB is a side of the required regular inscribed decagon.

Proof.—Join BX and BO.

Substituting AB for its equal OX we have

$$\frac{OA}{AB}=\frac{AB}{AX}.$$

Hence triangles AOB and ABX are similar. § 285

[Having the angle A common and the including sides proportional.]

But AOB is isosceles. § 150

Therefore ABX is isosceles, and $AB=BX=OX$. Cons.

Whence OXB is isosceles, and angle y=angle x. § 71

Then angle $z=x+y=2x$. § 59

And angle $OBA=A=z=2x$. § 71

Hence, in the triangle AOB,

angle $OAB+OBA+x=5x=2$ right angles. § 58

Therefore $x=\frac{1}{5}$ of 2 right angles, or $\frac{1}{10}$ of 4 right angles.

And arc $AB=\frac{1}{10}$ of the circumference. § 191

Therefore chord AB=side of regular inscribed decagon.

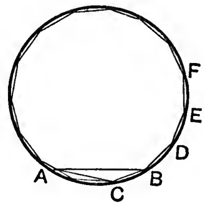
§ 469 I

Q. E. D.

476. *Exercise.*—Show that a regular pentagon is inscribed by joining the alternate vertices, A, C, E, G, I.

477. *Remark.*—A regular polygon of twenty sides is inscribed by bisecting the arcs AB, BC, etc., and, by continuing the process regular polygons of forty, eighty, etc., sides can be inscribed.

478. CONSTRUCTION. *To inscribe a regular pentedecagon in a given circle.*



GIVEN a circle AF.

TO CONSTRUCT—a regular inscribed pentedecagon.

Draw chord AB, the side of a regular inscribed hexagon. § 472

Draw chord AC, the side of a regular inscribed decagon. § 475

Then chord BC is a side of the required regular inscribed pentedecagon.

Proof: Arc AB is $\frac{1}{6}$ of the circumference.

Arc AC is $\frac{1}{10}$ of the circumference.

Hence Arc BC is $\frac{1}{6}-\frac{1}{10}$, or $\frac{1}{15}$ of the circumference.

Hence chord BC is the side of a regular inscribed polygon of fifteen sides. § 469 I

Q. E. D.

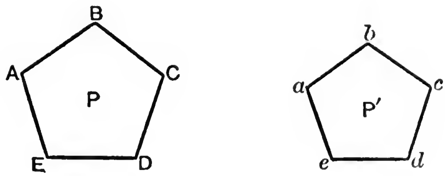
479. *Remark.*—A regular polygon of thirty sides can be inscribed by bisecting the arcs CB, BD, etc.; and, by continuing the process, regular polygons of sixty, one hundred and twenty, etc., sides can be inscribed.*

* We have seen how to inscribe polygons of

3, 6, 12, 24, 48, 96, etc., sides,
4, 8, 16, 32, 64, 128, etc., sides,
5, 10, 20, 40, 80, 160, etc., sides,
15, 30, 60, 120, 240, 480, etc., sides.

PROPOSITION III. THEOREM

480. *Two regular polygons of the same number of sides are similar.*



GIVEN—P and P', two regular polygons, each having n sides.

TO PROVE P and P' are similar.

$$\left.\begin{aligned} AB = BC = CD &= \text{etc.} \\ ab = bc = cd &= \text{etc.} \end{aligned}\right\} \qquad \S\ 461\ \text{I}$$

Dividing, $$\frac{AB}{ab} = \frac{BC}{bc} = \frac{CD}{cd} = \text{etc.}$$

That is, the two polygons have their homologous sides proportional.

Also, since there are n angles in each polygon, each angle of either polygon contains $\frac{2n-4}{n}$ right angles. § 463

That is, the two polygons are mutually equiangular.

Therefore they are similar. § 274

Q. E. D.

Up to the year 1796 these were the only regular polygons for which constructions were known. In that year Gauss, the greatest mathematician of the nineteenth century, then nineteen years of age, discovered a method of constructing, by means of ruler and compasses, a regular polygon of 17 sides, and in general all polygons of $2^m (2^n + 1)$ sides, m and n being integers, and $(2^n + 1)$ a prime number. This method was given in the *Disquisitiones Arithmeticæ*, published in 1801. In connection with this method Gauss enunciated the celebrated theorem that only a limited class of regular polygons are constructible by ruler and compass.

PROPOSITION IV. THEOREM

481. *In two regular polygons of the same number of sides, two corresponding sides are to each other as the radii or as the apothems.*

GIVEN—AB and $A'B'$, sides of regular polygons, each having the same number (n) of sides; and OA, $O'A'$, and OF, $O'F'$, the radii and apothems respectively.

TO PROVE $$\frac{AB}{A'B'}=\frac{OA}{O'A'}=\frac{OF}{O'F'}.$$

In the triangles OAB and $O'A'B'$,

angle O = angle O'. § 466

[Each being one-n^{th} of four right angles.]

Also $OA = OB$ § 150

and $O'A' = O'B'$.

Whence $$\frac{OA}{O'A'}=\frac{OB}{O'B'}.$$

Therefore the triangles are similar. § 285

Hence $$\frac{AB}{A'B'}=\frac{OA}{O'A'}.$$ § 274

And $$\frac{AB}{A'B'}=\frac{OF}{O'F'}.$$ § 290

Q. E. D.

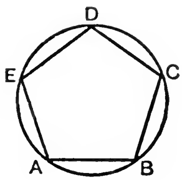
482. COR. I. *The perimeters of two regular polygons of the same number of sides are to each other as their radii or as their apothems.*

Hint.—Apply § 308.

483. COR. II. *The areas of two regular polygons of the same number of sides are to each other as the squares of their radii or as the squares of their apothems.*

PROPOSITION V. THEOREM

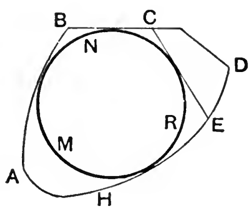
484. *The circumference of a circle is greater than the perimeter of an inscribed polygon.*



The proof is left to the student.

PROPOSITION VI. THEOREM

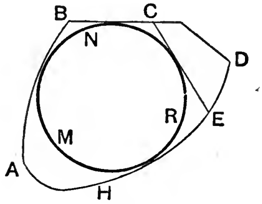
485. *The circumference of a circle is less than the perimeter of a circumscribed polygon or any enveloping line.*



GIVEN the circumference *MNR*.

TO PROVE—it is less than *ABCDEH*, any enveloping line.

Of all the lines enclosing the area *MNR* (of which the circumference *MNR* is one) there must be at least one *shortest* or *minimum* line.



The enveloping line *ABCDEH* is not a minimum line, since we can obtain a shorter one by drawing a tangent *CE*.

For $CE < CDE$. § 7

Therefore $ABCEH < ABCDEH$. Ax. 4

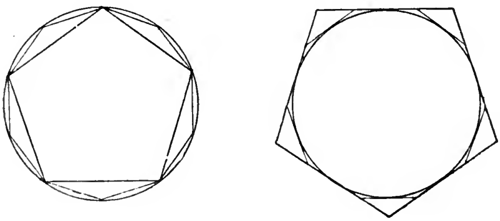
Likewise we may prove that *every* line enclosing *MNR* *except* the circumference is not minimum.

There remains therefore the circumference as the only minimum line. Q. E. D.

PROPOSITION VII. THEOREM

486. I. *If one regular inscribed polygon has twice as many sides as another, its perimeter and area are greater than those of the other.*

II. *If one regular circumscribed polygon has twice as many sides as another, its perimeter and area are less than those of the other.*



The proof is left to the student.

487. THEOREM. *If a variable x can be made less than any assigned quantity, the product of that variable and a decreasing quantity h can be made less than any assigned quantity.*

Let k be a constant greater than any value of h.

It has been proved that kx can be made less than any assigned quantity. § 187

But hx is always less than kx. Ax. 7

Hence hx can be made less than any assigned quantity.

488. COR. *If a variable x can be made less than any assigned quantity, then x^2 can be made less than any assigned quantity.*

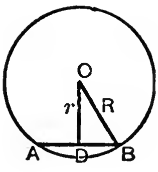
Hint.—Put x for h in the last theorem.

PROPOSITION VIII. LEMMA

489. *By doubling indefinitely the number of sides of a regular polygon inscribed in a given circle:*

I. *The apothem can be made to differ from the radius by less than any assigned quantity.*

II. *The square of the apothem can be made to differ from the square of the radius by less than any assigned quantity.*

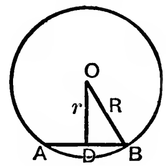


GIVEN—AB a side and r the apothem of a regular polygon inscribed in a circle whose radius is R.

TO PROVE—I. $R-r$ can be made as small as we please.

II. R^2-r^2 can be made as small as we please.

I. By doubling indefinitely the number of divisions of the circumference, the arc AB can be made as small as we please.



Therefore the chord AB, which is always less than the arc, can be made as small as we please.

Therefore DB, half of that chord, can be made as small as we please.

But $$R-r<DB.$$ § 137

Therefore $R-r$, which is always less than DB, can be made as small as we please. Q. E. D.

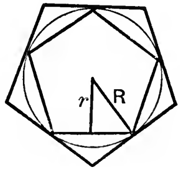
II. Since we can make DB as small as we please, we can also make $\overline{DB}^2$ as small as we please. § 488

But $$R^2-r^2=\overline{DB}^2.$$ § 318

Therefore we can make R^2-r^2, the equal of $\overline{DB}^2$, as small as we please. Q. E. D.

PROPOSITION IX. THEOREM

490. *The circumference of a circle is the limit which the perimeters of regular inscribed and circumscribed polygons approach when the number of their sides is doubled indefinitely; and the area of the circle is the limit of the areas of these polygons.*



GIVEN—P and p the perimeters, R and r the apothems, S and s the areas, respectively, of regular circumscribed and inscribed polygons of the same number of sides.

TO PROVE—I. The circumference of the circle is the common limit of P and p, when the number of sides is doubled indefinitely.

II. The area of the circle is the common limit of S and s, when the number of sides is doubled indefinitely.

I. Since the two regular polygons have the same number of sides, $\frac{P}{p}=\frac{R}{r}$. § 482

By division $\frac{P-p}{P}=\frac{R-r}{R}$. § 260

Or $P-p=P\frac{R-r}{R}$.

But, by doubling indefinitely the number of sides, $R-r$ can be made as small as we please. § 489 I

Hence $\frac{R-r}{R}$, the preceding variable divided by R, a constant quantity, can be made as small as we please. § 188

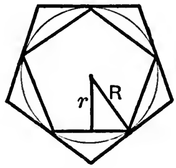
Hence $P\frac{R-r}{R}$, the preceding multiplied by P, a *decreasing* quantity (§ 486 II.), can be made as small as we please. § 487

Hence its equal $P-p$ can be made as small as we please.

But the circumference is always intermediate between P and p. §§ 484, 485

Therefore P and p, which can be made to differ *from each other* by less than any assigned quantity, can each be made to differ from the *intermediate quantity*, the circumference, by less than any assigned quantity.

But P and p can never equal the circumference. §§ 484, 485



Therefore by the definition of a limit the circumference is the common limit of P and p. § 185

Q. E. D.

II. Also, since the polygons are similar, § 480

$$\frac{S}{s} = \frac{R^2}{r^2}. \quad \text{§ 483}$$

By division $$\frac{S-s}{S} = \frac{R^2-r^2}{R^2}.$$

Or $$S-s = S\frac{R^2-r^2}{R^2}.$$

But R^2-r^2 can be made as small as we please. § 489 II

Hence $\frac{R^2-r^2}{R^2}$, the preceding variable divided by R^2, a constant quantity, can be made as small as we please. § 188

Hence $S\frac{R^2-r^2}{R^2}$, the preceding multiplied by S, a *decreasing* quantity (§ 486 II.), can be made as small as we please. § 487

Hence its equal $S-s$ can be made as small as we please.

But the area of the circle is always intermediate between S and s. Ax. 10

Therefore S and s, which can be made to differ *from each other* by less than any assigned quantity, can each be made to differ from the *intermediate quantity*, the area of the circle, by less than any assigned quantity.

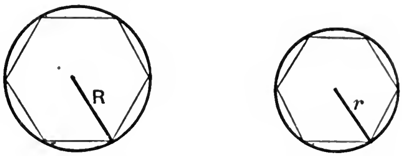
But S and s can never equal the area of the circle. Ax. 10

Therefore by the definition of a limit the area of the circle is the common limit of S and s. § 185

Q. E. D.

PROPOSITION X. THEOREM

491. *The ratio of the circumference of a circle to its diameter is the same for all circles.*



GIVEN—any two circles with radii R and r, and circumferences C and c respectively.

TO PROVE $$\frac{C}{2R}=\frac{c}{2r}.$$

Inscribe in the two circles regular polygons of the same number of sides, and call their perimeters P and p.

Then $$\frac{P}{p}=\frac{R}{r}=\frac{2R}{2r}.$$ § 482

Hence $$\frac{P}{2R}=\frac{p}{2r}.$$ § 256

As the number of sides of the two inscribed polygons is indefinitely doubled, P approaches C as its limit and p approaches c as its limit. § 490

Hence $\frac{P}{2R}$ approaches $\frac{C}{2R}$ as its limit,

and $\frac{p}{2r}$ approaches $\frac{c}{2r}$ as its limit. § 190

But always $$\frac{P}{2R}=\frac{p}{2r}.$$

Hence $$\frac{C}{2R}=\frac{c}{2r}.$$ § 186

Q. E. D.

492. *Def.*—This uniform ratio of a circumference to its diameter is called π. It will be shown in § 502 that its value is approximately $3\frac{1}{7}$.

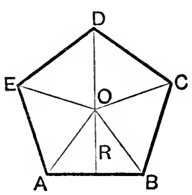
493. COR. *The circumference of a circle is equal to its radius multiplied by* 2π.

Hint.—By definition $\frac{C}{2R} = \pi$.

494. *Exercise.*—The radius of a locomotive driving-wheel is 6 feet; how far does it roll on the track in one revolution?

PROPOSITION XI. THEOREM

495. *The area of a regular polygon is equal to half the product of its apothem and perimeter.*



GIVEN—a regular polygon $ABCDE$, R its apothem, and P its perimeter.

TO PROVE area polygon $= \frac{1}{2} R \times P$.

Draw from O the centre OA, OB, OC, etc.

The polygon is thus divided into as many triangles as it has sides.

The apothem R is their common altitude, and their bases are the sides of the polygon.

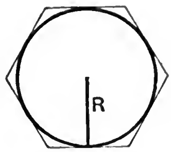
The area of *each* is $\frac{1}{2} R$ times its base. § 390

The area of *all* is $\frac{1}{2} R$ times the sum of their bases.

Or area polygon $= \frac{1}{2} R \times P$. Q. E. D.

PROPOSITION XII. THEOREM

496. *The area of a circle equals half the product of its radius and circumference.*



GIVEN—a circle with radius R, circumference C, and area S.

TO PROVE $S = \frac{1}{2} R \times C$.

Circumscribe a regular polygon and call its perimeter C' and area S'.

Then $S' = \frac{1}{2} R \times C'$. § 495

[The area of a regular polygon equals half the product of its apothem and perimeter.]

Let the number of sides of the regular circumscribed polygon be indefinitely increased.

C', the perimeter of the polygon, approaches C, the circumference, as its limit. § 490

Hence $\frac{1}{2} R \times C'$ approaches $\frac{1}{2} R \times C$ as its limit. § 189

Also S' approaches S as its limit. § 490

But *always* $S' = \frac{1}{2} R \times C'$.

Therefore $S = \frac{1}{2} R \times C$. § 186

Q. E. D.

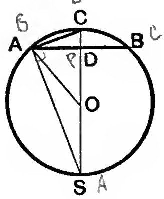
497. COR. I. *The area of a circle is* πR^2.

498. COR. II. *The area of a sector whose angle is* $n°$, *is* $\frac{n}{360}(\pi R^2)$.

499. COR. III. *The areas of two circles are to each other as the squares of their radii, or as the squares of their diameters.*

PROPOSITION XIII. PROBLEM

500. *Given a circle of unit diameter and the side of a regular inscribed polygon, to find the side of a regular inscribed polygon of double the number of sides.*



GIVEN—the circle O of unit diameter, and AB, or s, the side of a regular inscribed polygon.

TO FIND—the length of AC, or x, a side of a regular polygon of double the number of sides.

Draw CS, the diameter perpendicular to AB.

Join AO and AS.

Now CAS is a right angle. § 202

And $AD = \frac{s}{2}$. § 167

Also $CS = 1,\ AO = \frac{1}{2},\ CO = \frac{1}{2}$. Cons.

Hence $\overline{AC}^2 = CS \times CD$ § 312

$$= 1 \times CD = CD = CO - DO = \frac{1}{2} - DO$$

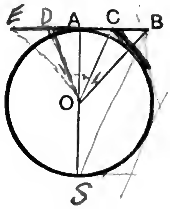
$$= \frac{1}{2} - \sqrt{\overline{AO}^2 - \overline{AD}^2}$$ § 318

$$= \frac{1}{2} - \sqrt{\left(\frac{1}{2}\right)^2 - \left(\frac{s}{2}\right)^2} = \frac{1 - \sqrt{1 - s^2}}{2}.$$

Therefore $$AC = x = \sqrt{\frac{1 - \sqrt{1 - s^2}}{2}}.$$

PROPOSITION XIV. PROBLEM

501. *Given a circle of unit diameter and the side of a regular circumscribed polygon, to find the side of a regular circumscribed polygon of double the number of sides.*



GIVEN—the circle O of unit diameter and AB, or $\frac{s}{2}$, *half* the side of a regular circumscribed polygon.

TO FIND—AC, or $\frac{x}{2}$, *half* the side of a regular circumscribed polygon of double the number of sides.

Join OA, OC, OB.

Angle AOB is *half* the angle between successive radii of the first polygon. § 468

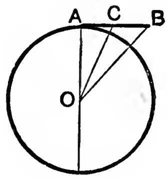
Angle AOC is *half* the angle between successive radii of the second polygon. § 468

But the angle between successive radii in the second polygon is half that in the first. § 466

Therefore angle $AOC=\frac{1}{2}$ angle AOB, that is, OC bisects the angle AOB.

Hence $$\frac{AC}{CB}=\frac{AO}{OB},$$ § 327

or $$\frac{AC}{AB-AC}=\frac{AO}{\sqrt{\overline{AO}^2+\overline{AB}^2}}.$$



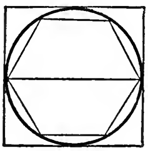
Substituting, $$\frac{\frac{x}{2}}{\frac{s}{2}-\frac{x}{2}}=\frac{\frac{1}{2}}{\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{s}{2}\right)^2}}.$$

Simplifying, $$\frac{x}{s-x}=\frac{1}{\sqrt{1+s^2}}.$$

Solving, $$x=\frac{s}{1+\sqrt{1+s^2}}.$$

PROPOSITION XV. PROBLEM

502. *To compute the ratio of the circumference of a circle to its diameter approximately.*



GIVEN a circle.

TO FIND—the ratio of its circumference to its diameter approximately, or the value of π.

Since the ratio π is the same for all circles (§ 491), it is sufficient to compute it for any one.

We select a circle of which the diameter is unity.

The radius of this circle will be $\frac{1}{2}$ and the side of a regular inscribed hexagon will be $\frac{1}{2}$; and of a circumscribed square 1.

Using the formula $x=\sqrt{\frac{1-\sqrt{1-s^2}}{2}}$ (§ 500), we form the following table giving the length of the sides of regular inscribed polygons of 6, 12, 24, etc., sides. The length of the perimeter is obtained by multiplying the length of one side by the number of sides.

INSCRIBED REGULAR POLYGONS

NO. SIDES	LENGTH OF SIDE	LENGTH OF PERIMETER
6	0.500000	3.000000
12	0.258819	3.105829
24	0.130526	3.132629
48	0.065403	3.139350
96	0.032719	3.141032
192	0.016362	3.141453
384	0.008181	3.141558

Using the formula $x=\frac{s}{1+\sqrt{1+s^2}}$ (§ 501), we form the following table giving the length of the sides and perimeters of regular circumscribed polygons of 4, 8, 16, etc., sides.

CIRCUMSCRIBED REGULAR POLYGONS

NO. SIDES	LENGTH OF SIDE	LENGTH OF PERIMETER
4	1.000000	4.000000
8	0.414214	3.313709
16	0.198912	3.182598
32	0.098492	3.151725
64	0.049127	3.144118
128	0.024549	3.142224
256	0.012272	3.141750
512	0.006136	3.141632

But the length of the circumference must be intermediate between the lengths of the circumscribed and inscribed poly gons. Hence it must be intermediate between 3.141558 and 3.141632. Hence 3.1416 is the nearest approximation to four decimal places.

Since the diameter of the circle is 1, the ratio of the circumference to the diameter is $\frac{3.1416}{1}$, or 3.1416.

That is, $\pi = 3.1416.$*

503. *Exercise.*—By means of the value of π just found and the formulas for the circumference and area of a circle, find the circumference and area of a circle whose radius is 23.16 inches.

* The earliest known attempt to obtain the area of the circle or to "square the circle" is recorded in a MS. in the British Museum recently deciphered. It was written by an Egyptian priest, *Ahmes*, at least as early as 1700 B.C., and possibly several centuries earlier. The method was to *deduct from the diameter of the circle one-ninth of itself and square the remainder.* This is equivalent to using a value of π equal to 3.16. *Archimedes* (about 250 B.C.), the greatest mathematician of ancient times, proved, by methods essentially the same as those employed in the text, that the true value of π lies between $3\frac{1}{7}$ and $3\frac{10}{71}$, i. e., between 3.1429 and 3.1408. *Ptolemy* (about 150 A.D.) used the value 3.1417. In the 16th century *Metrus*, of Holland, using polygons up to 1536 sides, obtained the easily-remembered approximation $\frac{355}{113}$ (write 113355 and divide last three by first three), which is correct to six places of decimals. *Romanus*, also of Holland, using polygons of 1,073,741,324 sides, soon after computed sixteen places. With the better methods of higher mathematics various mathematicians have extended the computations gradually, until *Mr. Shanks*, in 1873, published a result to 707 places, the first 411 of which have been verified by *Dr. Rutherford.* The following are the first figures of his result:
π=3.141,592,653,589,793,238,462,643,383,279,502,884,197,169,399,375,105,8.
How accurate a value this is may be inferred from Prof. Newcomb's remark that *ten* decimals would be sufficient to calculate the circumference of the earth to a fraction of an inch if we had an exact knowledge of the diameter.

The Greeks sought in vain for a perfectly accurate result or geometrical construction for obtaining a square equivalent to the circle, as did many mediæval mathematicians. "Circle squarers" still exist among the ignorant, although *Lambert* (about A.D. 1750) proved π incommensurable, i. e., inexpressible as a finite fraction, and *Lindemann*, in 1882, proved it is also transcendental, *i.e.*, inexpressible as a radical or root of any algebraic equation with integral coefficients.

PROBLEMS OF DEMONSTRATION

504. The angle at the centre of a regular polygon is the supplement of any angle of the polygon.

505. If the sides of a regular circumscribed polygon are tangent to the circle at the vertices of the similar inscribed polygon, then each vertex of the circumscribed figure lies in the prolongation of the apothem of the inscribed.

506. If the sides of a regular circumscribed polygon are tangent to the circle at the middle points of the arcs subtended by the sides of a similar inscribed polygon, then the sides of the circumscribed figure are parallel to those of the inscribed, and the vertices lie in the prolongation of the radii.

507. If from any point within a regular polygon of n sides perpendiculars are drawn to the several sides, the sum of these perpendiculars is equal to n times the apothem.

Hint.—Apply § 495.

508. The area of a circumscribed square is double that of an inscribed square.

509. The side of an inscribed equilateral triangle is equal to one-half the side of a circumscribed equilateral triangle, and the area of the first is one-fourth that of the second.

510. The apothem of an inscribed equilateral triangle is equal to half the radius.

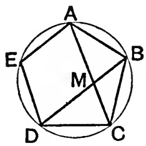
511. The apothem of a regular inscribed hexagon is equal to half the side of the inscribed equilateral triangle.

512. The radius of a regular inscribed polygon is a mean proportional between its apothem and the radius of the similar regular circumscribed polygon.

513. The area of the ring included between two concentric circles is equal to that of a circle whose radius is one half a chord of the outer circle drawn tangent to the inner.

514. In two circles of different radii, angles at the centre subtended by arcs of equal length are to each other inversely as their radii.

515. Two diagonals of a regular pentagon, not drawn from a common vertex, divide each other in extreme and mean ratio.



Hint.—Prove the triangles ABC and BCM similar (§ 275). Then prove $AM = AB = BC$ (§ 77), and substitute in the proportion derived from the first step.

PROBLEMS OF CONSTRUCTION

516. Having given a circle, to construct the circumscribed hexagon, octagon, and decagon.

517. Upon a given straight line as a side to construct a regular hexagon.

518. Having given a circle and its centre, to find two opposite points in the circumference by means of compasses only.

519. To divide a right angle into five equal parts.

520. To inscribe a square in a given quadrant.

521. Having given two circles, to construct a third circle equivalent to their difference.

522. To divide a circle into any number of equivalent parts by circumferences concentric with it.

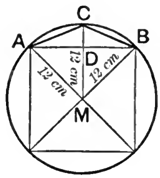
PROBLEMS FOR COMPUTATION

523. (1.) Find the number of degrees in an angle of each of the following regular polygons: (*a*) triangle, (*b*) pentagon, (*c*) hexagon, (*d*) octagon, and (*e*) decagon.

(2.) What is the area of a regular pentagon inscribed in a circle whose radius is 12 cm.?

(3.) If the side of a regular hexagon is 10 m., find the number of square feet in its area.

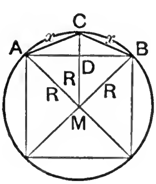
(4.) Find the area of a regular octagon inscribed in a circle whose radius is 12 cm.



(5.) If the radius of a circle is R, find the side and the apothem of a regular inscribed (*a*) triangle, (*b*) square, (*c*) hexagon.

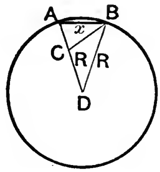
(6.) If, in the above example, $R = 15.762$, find the numerical value of the side and apothem for each of the three polygons.

(7.) Prove that the side of a regular octagon, inscribed in a circle whose radius is R, is equal to $R\sqrt{2-\sqrt{2}}$.



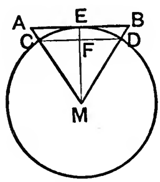
(8.) Find the apothem of a regular octagon inscribed in a circle whose radius is R.

(9.) If the radius of a circle is R, find the side of a regular inscribed decagon.



(10.) What is the apothem of the above decagon?

(11.) Find the side of a regular hexagon circumscribed about a circle whose radius is R.



(12.) If the radius of a circle is R, prove that the area of a regular inscribed dodecagon is $3R^2$.

(13.) There are three regular hexagons; the side of the first is 20 in., that of the second is 1 m., that of the third 5 ft. Find in meters the side of a fourth regular hexagon whose area is equal to the sum of the areas of the first three.

(14.) A wheel, having a radius of 1.5 ft., made 3360 revolutions in going over the road from one town to another. How many miles apart are the towns?

(15.) If the circumference of a circle is 50 in., find the radius.

(16.) If a wheel has 35 cogs, and the distance between the middle points of the cogs is 12 in., find the radius of the wheel.

(17.) Find the width of a ring of metal the outer circumference of which is 88 m. in length, and the inner circumference 66 m.

(18.) If the radius of a circle is 16 cm., how many degrees, minutes, and seconds are there in an arc 10 cm. long?

(19.) Find the number of feet in an arc of $20°$ if the radius of the circle is 12 m.

(20.) How many degrees are there in an arc whose length is equal to the radius of the circle?

(21.) If an arc of $30° = 12.5664$ in., find the radius of the circle.

(22.) If the radius of a circle is 15 cm., find the length of the arc subtended by a chord 15 cm. in length.

(23.) If the circumference of a circle is c, find its radius and diameter.

(24.) Find the area of a circle whose radius is (*a*) 11 in.; (*b*) 17.146 m.; (*c*) 35 ft.

(25.) Find the ratio of the areas of two circles if the radius of one is the diameter of the other.

(26.) If the circumference of a circle is 60 ft., find the area.

(27.) The radius of a circle is 13 in. Find the side of a square whose area is equal to that of the circle.

(28.) The side of an inscribed square is 23 m. What is the area of the circle?

(29.) What is the area of a circle inscribed in a square whose surface contains 211 ares?

(30.) Find the side of the largest square that can be cut from the cross-section of a tree 14 ft. in circumference.

(31.) If the diameter of a given circle is 5 cm., find the diameter of a circle one-fourth as large.

(32.) A rectangle and a circle have equal perimeters. Find the difference in their areas if the radius of the circle is 9 in. and the width of the rectangle is three-fourths its length.

(33.) If the radius of a circle is 25 m., what is the radius of a concentric circle which divides it into two equivalent parts?

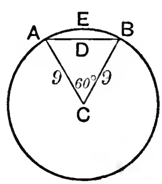
(34.) The radii of two concentric circles are respectively 9 and 6 in. Find the area of the ring bounded by their circumferences.

(35.) The chord of a segment of a circle is 34 in. in length, and the height of the segment is 8 in. Find the radius.

(36.) In a circle whose radius is 18 in., find the height of a segment whose chord is 28 in. in length.

(37.) If the radius of a circle is 16 cm., what is the area of a sector having an angle of 24°?

(38.) The radius of a circle is 9 in. Find the area of a segment whose arc is 60°.



Hint.—Area of segment $AEBD$ = area of sector $AEBC$ minus area of triangle ABC.

(39.) If the radius of a circle is R, find the area of the segment subtended by the side of a regular hexagon.

(40.) If the radius of a circle is R, find the area of a segment subtended by the side of (*a*) an inscribed equilateral triangle, (*b*) an inscribed regular octagon, (*c*) an inscribed regular decagon.

EXERCISES

BOOK I

PROBLEMS OF DEMONSTRATION

1. The bisector of an angle of a triangle is less than half the sum of the sides containing the angle.

2. The median drawn to any side of a triangle is less than half the sum of the other two sides, and greater than the excess of that half sum above half the third side.

3. The shortest of the medians of a triangle is the one drawn to the longest side.

4. The sum of the three medians of a triangle is less than the sum of the three sides, but greater than half their sum.

5. In any triangle the angle between the bisector of the angle opposite any side and the perpendicular from the opposite vertex on that side is equal to half the difference of the angles adjacent to that side.

6. LM and PR are two parallels which are cut obliquely by AB in the points A, B, and at right angles by AC in the points A, C; the line BED, which cuts AC in E and LM in D, is such that ED is equal to $2AB$. Prove that the angle DBC is one-third the angle ABC.

7. The sum of the diagonals of a quadrilateral is less than the sum of the four lines joining any point other than the intersection of the diagonals to the four vertics.

8. The difference between the acute angles of a right triangle is equal to the angle between the median and the perpendicular drawn from the vertex of the right angle to the hypotenuse.

9. In a right triangle the bisector of the right angle also bisects the angle between the perpendicular and the median from the vertex of the right angle to the hypotenuse.

10. In the triangle formed by the bisectors of the exterior angles of a given triangle, each angle is one-half the supplement of the opposite angle in the given triangle.

11. A right triangle can be divided into two isosceles triangles.

12. A median of a triangle is greater than, equal to, or less than half of the side which it bisects, according as the angle opposite that side is acute, right, or obtuse.

13. The point of intersection of the perpendiculars erected at the middle of each side of a triangle, the point of intersection of the three medians, and the point of intersection of the three perpendiculars from the vertices to the opposite sides are in a straight line; and the distance of the first point from the second is half the distance of the second from the third.

14. Find the locus of a point the sum or the difference of whose distances from two fixed straight lines is given.

15. On the side AB, produced if necessary, of a triangle ABC, AC' is taken equal to AC; similarly on AC, AB' is taken equal to AB, and the line $B'C'$ drawn to cut BC in P. Prove that the line AP bisects the angle BAC.

16. The point of intersection of the straight lines which join the middle points of opposite sides of a quadrilateral is the middle point of the straight line joining the middle points of the diagonals.

17. The angle between the bisector of an angle of a triangle and the bisector of an exterior angle at another vertex is equal to half the third angle of the triangle.

18. If L and M are the middle points of the sides AB, CD of a parallelogram $ABCD$, the straight lines, DL, BM trisect the diagonal AC.

19. ABC is an equilateral triangle; BD and CD are the bisectors of the angles at B and C. Prove that lines through D parallel to the sides AB and AC trisect BC.

20. The angle between the bisectors (produced only to their point of intersection) of two adjacent angles of a quadrilateral is equal to half the sum of the two other angles of the quadrilateral. The acute angle between the bisectors of two opposite angles of a quadrilateral is equal to half the difference of the other angles.

21. The bisectors of the angles of a quadrilateral form a second quadrilateral of which the opposite angles are supplementary. When the first quadrilateral is a parallelogram, the second is a rectangle whose diagonals are parallel to the sides of the parallelogram and each equal to the difference of two adjacent sides of the parallelogram. When the first quadrilateral is a rectangle, the second is a square.

22. Two quadrilaterals are equal if an angle of the one is equal to an angle of the other, and the four sides of the one are respectively equal to the four similarly situated sides of the other.

23. If two polygons have the same number of sides and this number is odd, and if one polygon can be placed upon the other so that the middle points of the sides of the first fall upon the middle points of the sides of the second, the polygons are equal.

PROBLEMS OF CONSTRUCTION

24. Find a point in a straight line such that the sum of its distances from two fixed points on the same side of the straight line shall be the least possible.

25. Find a point in a straight line such that the difference of its distances from two fixed points on opposite sides of the line shall be the greatest possible.

26. Draw through a given point within a given angle a straight line such that the part intercepted between the sides of the angle shall be bisected by the given point.

27. Through a given point without a straight line to draw a straight line making a given angle with the given line.

28. Divide a rectangle 7 in. long and 3 in. broad into three figures which can be joined together so as to form a square.

BOOK II

PROBLEMS OF DEMONSTRATION

29. If a circle is circumscribed about an equilateral triangle and from any point in the circumference straight lines are drawn to the three vertices, one of these lines is equal to the sum of the other two.

30. If one circle touches another internally at P and a tangent to the first at Q intersects the second in M, N, then the angles MPQ, NPQ are equal.

31. The centre of one circle is on the circumference of another; if A and B are the points in which the common tangents touch the second, prove that the line AB is tangent to the first.

32. The trapezoid of which the non-parallel sides are equal is the only trapezoid which can be inscribed in a circle.

33. From any point on the circumference of a circle circumscribed about an equilateral triangle ABC, straight lines are drawn parallel respectively to BC, CA, AB, meeting the sides CA, AB, BC at M, N, O. Prove that M, N, O are in the same straight line.

34. If a quadrilateral be inscribed in a circle and the opposite sides produced to meet at M and N, prove that the bisectors of the angles at M and N meet at right angles.

35. Two circles pass through the vertex and a point in the bisector of an angle. Prove that the portions of the sides of the angle intercepted between their circumferences are equal.

36. Each angle formed by joining the feet of the perpendiculars of a triangle is bisected by the perpendicular from the opposite vertex.

37. Circumscribe a circle about a triangle; from one vertex drop a perpendicular on the opposite side to meet it in M, and produce to meet the circumference in N. Then, if P is the intersection of the perpendiculars, $PM = MN$.

38. A fixed circle touches a fixed straight line; any circle is drawn touching the fixed circle at B and the fixed straight line at C. Prove that the straight line BC passes through a fixed point.

39. The distance from the centre of the circle circumscribed about a triangle to a side is equal to half the distance from the opposite vertex to the intersection of the three perpendiculars from the vertices to the sides.

40. Prove that the straight lines joining the vertices of a triangle with the opposite points of tangency of the inscribed circle meet in a point.

41. If two points are given on the circumference of a given circle, another fixed circle can be found such that if any two lines be drawn from the given points to intersect on its circumference, the straight line joining the points in which these lines meet the given circle a second time will be of constant length.

42. If the three diagonals joining the opposite vertices of a hexagon are equal and the opposite sides are parallel in pairs, the hexagon can be inscribed in a circle.

43. Equilateral triangles are constructed on the sides of a given triangle and external to it. Prove that the three lines, each joining the outer vertex of one of the equilateral triangles to the opposite vertex of the given triangle, meet in a point and are equal.

44. On each side of a triangle construct an isosceles triangle with the adjacent angles equal to 30°. Prove that the straight lines joining the outer vertices of these three triangles are equal.

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45. One side and the opposite angle of a triangle are given, and equilateral triangles are constructed on the other two (variable) sides. Find the locus of the middle point of the straight line joining the outer vertices of the equilateral triangles.

46. Through a vertex of an equilateral triangle is drawn any straight line PQ, terminated by the perpendiculars to the opposite side erected at the extremities of that side; on PQ as a side a second equilateral triangle is constructed. Find the locus of the opposite vertex of the second equilateral triangle.

47. The sides of a right triangle are given in position, its hypotenuse in length. Find the locus of the middle point of the hypotenuse.

48. AC, BD, are fixed diameters of a circle, at right angles to each other, and P is any point on the circumference. PA cuts BD in E; EF, parallel to AC, cuts PB in F. Prove that the locus of F is a straight line.

PROBLEMS OF CONSTRUCTION

49. Draw four circles through a given point and tangent to two given circles.

50. Through a given point draw a straight line cutting a given straight line and a given circle, such that the part of the line between the point and the given line may be equal to the part within the given circle.

51. Find a point in a given straight line such that tangents from it to two given circles shall be equal.

52. Construct a right triangle, having given one side and the perpendicular from the vertex of the right angle on the hypotenuse.

53. The distances from a point to the three nearest corners of a square are 1 in., 2 in., $2\frac{1}{2}$ in. Construct the square.

54. Construct a right triangle, having given the medians from the extremities of the hypotenuse.

55. Construct a right triangle, having given the difference between the hypotenuse and each side.

56. Construct a triangle, having given one angle and the medians drawn from the vertices of the other angles.

57. Construct a triangle, having given an angle, the perpendicular from its vertex on the opposite side, and the sum of the sides including that angle.

58. Having given two concentric circles, draw a chord of the larger circle, which shall be divided into three equal parts by the circumference of the smaller circle.

59. Inscribe in a circle a quadrilateral $ABCD$, having the diagonal AC given in direction, the diagonal BD in magnitude, and having given the position of the point E in which the sides AB and CD meet when produced.

60. Draw a chord of given length through a given point, within or without a given circle.

61. Construct an equilateral triangle such that one vertex is at a given point, and the other two vertices are on a given straight line and a given circumference respectively.

BOOK III

PROBLEMS OF DEMONSTRATION

62. If from a given point straight lines are drawn to the extremities of any diameter of a given circle, the sum of the squares of these lines will be constant.

63. The straight line joining the middle of the base of a triangle to the middle point of the line drawn from the opposite vertex to the point at which the inscribed circle touches the base, passes through the centre of the inscribed circle.

64. The square of the straight line joining the centre of a circle to to any point of a chord plus the product of the segments of the chord is equal to the square of the radius.

65. P and Q are two points on the circumscribing circle of the triangle ABC, such that the distance of either point from A is a mean proportional between its distances from B and C. Prove that the difference between the angles PAB, QAC is half the difference between the angles ABC, ACB.

66. If a quadrilateral be circumscribed about a circle, prove that the middle points of its diagonals and the centre of the circle are in a straight line.

67. From the vertex of the right angle C of a right triangle ACB straight lines CD and CE are drawn, making the angles ACD, ACE each equal to the angle B, and meeting the hypotenuse in D and E. Prove that $$\overline{DC}^2 : \overline{DB}^2 = AE : EB.$$

68. $ABCD$ is a parallelogram; the circle through A, B, and C cuts AD in A', and DC in C'. Prove that
$$A'D : A'C' = A'C : A'B.$$

69. If two intersecting chords are drawn in a semicircle from the extremities of the diameter, the sum of the products of the segment adjacent to the diameter in each by the whole chord is equal to the square of the diameter.

70. If a quadrilateral circumscribe a circle the two diagonals and the two lines joining the points where the opposite sides of the quadrilateral touch the circle will all four meet in a point.

71. There are two points whose distances from three fixed points are in the ratios $p:q:r$. Prove that the straight line joining them passes through a fixed point whatever be the values of the ratios.

72. The lines joining the vertices of an equilateral triangle ABC to any point D meet the circumscribing circle in the points A', B', C'. Prove that $AD.\ AA'+BD.\ BB'+CD.\ CC'=2\overline{AB}^2$.

73. If from any point perpendiculars are drawn to all the sides of a polygon, the two sums of the squares of the alternate segments of the sides are equal.

74. One circle touches another internally, and a third circle whose radius is a mean proportional between their radii passes through the point of contact. Prove that the other intersections of the third circle with the first two are in a line parallel to the common tangent of the first two.

75. If two tangents are drawn to a circle at the extremities of a diameter, the portion of any third tangent intercepted between them is divided at its point of contact into segments whose product is equal to the square of the radius.

76. A straight line AB is divided harmonically at P and Q; M, N are the middle points of AB and PQ. If X be any point on the line, prove that $XA.\ XB+XP.\ XQ=2XM.\ XN$.

77. The radius of a circle drawn through the centres of the inscribed and any two escribed circles of a triangle is double the radius of the circumscribed circle of the triangle.

78. The centres of the four escribed circles of a quadrilateral lie on the circumference of a circle.

79. O, O_1, O_2, O_3 are the centres of the inscribed and three escribed circles of a triangle ABC. Prove that

$$AO.\ AO_1.\ AO_2.\ AO_3=\overline{AB}^2.\ \overline{AC}^2.$$

80. The opposite sides of a quadrilateral inscribed in a circle, when produced, meet at P and Q; prove that the square of PQ is equal to the sum of the squares of the tangents from P and Q to the circle.

LOCI

81. A is a point on the circumference of a given circle, P a point without the circle. AP cuts the circle again in B, and the ratio AP : AB is constant.

Find the locus of P.

82. Find the locus of a point whose distances from two given points are in a given ratio.

83. Find the locus of a point the sum of the squares of whose distances from the vertices of a given triangle is constant.

PROBLEMS OF CONSTRUCTION

84. Draw a circle such that, if straight lines be drawn from any point in its circumference to two given points, these lines shall have a given ratio.

85. Construct a triangle, having given the base, the line bisecting the opposite angle, and the diameter of the circumscribed circle.

86. Construct a right triangle, having given the difference between the sides and the difference between the hypotenuse and one side.

87. Construct a triangle, having given the perimeter, the altitude, and that one base angle is twice the other.

88. Construct a triangle, having given an angle, the length of its bisector, and the sum of the including sides.

89. From one extremity of a diameter of a given circle draw a straight line such that the part intercepted between the circumference and the tangent at the other extremity shall be of given length.

90. Divide a semi-circumference into two parts such that the radius shall be a mean proportional between the chords of the parts.

91. Construct a triangle, similar to a given triangle, such that two of its vertices may be on lines given in position, and its third vertex be at a given point.

92. Through four given points draw lines which will form a quadrilateral similar to a given quadrilateral.

93. Find a point such that its distances from three given points may have given ratios.

94. Divide a straight line harmonically in a given ratio.

95. A line perpendicular to the bisector of an angle of a triangle is drawn through the point in which the bisector meets the opposite side. Prove that the segment on either of the other sides between this line and the vertex is a harmonic mean between those sides.

96. Draw through a given point within a circle a chord which shall be divided at that point in mean and extreme ratio.

PROBLEMS FOR COMPUTATION

97. (1.) The sides of a right triangle are 15 ft. and 18 ft. The hypotenuse of a similar triangle is 20 ft. Find its sides.

(2.) The sides of a right triangle are 16.213 in. and 32.426 in. Find the ratio of the segments of the hypotenuse formed by the altitude upon the hypotenuse.

(3.) In an isosceles triangle the vertex angle is 45°; each of the equal sides is 16 yds. Find the base in meters.

(4.) In a triangle whose sides are 247.93 mm., 641.98 mm., 521.23 mm., find the altitude upon the shortest side.

(5.) In a triangle whose sides are 4, 7, and 9, find the median drawn to the shortest side.

(6.) In a triangle whose sides are 123.41 in., 246.93 in., 157.62 in., compute the bisector of the largest angle.

(7.) Two adjacent sides of a parallelogram are 49 cm. and 53 cm. One diagonal is 58 cm. Find the other diagonal.

(8.) If the chord of an arc is 720 ft., and the chord of its half is 376 ft., what is the diameter of the circle?

(9.) From a point without a circle two tangents are drawn making an angle of 60°. The length of each tangent is 15 in. Find the diameter of the circle.

(10.) Find the radius of a circle circumscribing a triangle whose sides are 35.421 cm., 36.217 cm., 423.92 cm.

BOOK IV

PROBLEMS OF DEMONSTRATION

98. A straight line AB is bisected in C and divided unequally in D. Prove that the sum of the squares on AD and DB is equal to twice the sum of the squares on AC and CD.

99. The area of a triangle is equal to the product of its three sides divided by four times the radius of its circumscribed circle.

100. Prove, by a geometrical construction, that the square on the hypotenuse of a right triangle is equal to four times the triangle plus the square on the difference of the sides.

101. Prove, by a geometrical construction, that the square on the hypotenuse of a right triangle is equal to the square on the sum of the sides minus four times the triangle.

102. On the side BC of the rectangle $ABCD$ as diameter describe a circle. From its centre E draw the radius EG parallel to CD and in the direction C to D. Join G and C by a straight line cutting the diagonal BD in H. From H draw the line HK parallel to CD and in the direction C to D, cutting the circumference of the circle in K. Join BK and produce to meet CD in L. Then CL is the side of a square which is equivalent to the rectangle $ABCD$.

103. Construct any parallelograms $ACDE$ and $BCFG$ on the sides AC and BC of a triangle and exterior to the triangle. Produce ED and GF to meet in H and join HC; through A and B draw AL and BM equal and parallel to HC. Prove that the parallelogram $ALMB$ is equal to the sum of the parallelograms which have been constructed on the sides.

104. If similar triangles be circumscribed about and inscribed in a given triangle, the area of the given triangle is a mean proportional between the areas of the inscribed and circumscribed triangles.

105. Any fourth point P is taken on the circumference of a circle through A, B, and C. Prove that the middle points of PA, PB, PC form a triangle similar to the triangle ABC, of one-fourth the area, and such that its circumscribing circle always touches the given circle at P.

106. Equilateral triangles are constructed on the four sides of a square all lying within the square. Prove that the area of the star-shaped figure formed by joining the vertex of each triangle to the two nearest corners of the square is equal to eight times the area of one of the equilateral triangles minus three times the area of the square.

107. A hexagon has its three pairs of opposite sides parallel. Prove that the two triangles which can be formed by joining alternate vertices are of equal area.

108. A quadrilateral and a triangle are such that two of the sides of the triangle are equal to the two diagonals of the quadrilateral and the angle between these sides is equal to the angle between the diagonals. Prove the areas of the quadrilateral and triangle are equal.

109. Prove that the straight lines drawn from the corners of a square to the middle points of the opposite sides taken in order form a square of one-fifth the area of the original square.

110. The area of the octagon formed by the straight lines joining each vertex of a parallelogram to the middle points of the two opposite sides is one-sixth the area of the parallelogram.

111. $ABCD$ is a parallelogram. A point E is taken on CD such that CE is an n^{th} part of CD; the diagonal AC cuts BE in F. Prove the following continued proportion connecting the areas of the parts of the parallelogram

$$ADEFA : AFB : BFC : CFE = n^2 + n - 1 : n^2 : n : 1$$

112. The squares $ACKE$ and $BCID$ are constructed on the sides of a right triangle ABC; the lines AD and BE intersect at G; AD cuts CB in H, and BE cuts AC in F. Prove that the quadrilateral $FCHG$ and the triangle ABG are equivalent.

PROBLEMS OF CONSTRUCTION

113. Construct an equilateral triangle which shall be equal in area to a given parallelogram.

114. Construct a square which shall have a given ratio to a given square.

115. A pavement is made of black and white tiles, the black being squares, the white equilateral triangles whose sides are equal to the sides of the squares. Construct the pattern so that the areas of black and white may be in the ratio $\sqrt{3} : 4$.

116. Produce a given straight line so that the square on the whole line shall have a given ratio to the rectangle contained by the given line and its extension. When is the problem impossible?

117. Find a point in the base produced of a triangle such that a straight line drawn through it cutting a given area from the triangle may be divided by the sides of the triangle into segments having a given ratio.

118. Bisect a given quadrilateral by a straight line drawn through a vertex.

PROBLEMS FOR COMPUTATION

119. (1.) If the area of an equilateral triangle is 164.51 sq. in., find its perimeter.

(2.) The perimeter of an equilateral pentagon is 25.135 ft. Its area is 23.624 sq. ft. Find the area of a similar pentagon one of whose sides is 10.361 ft.

(3.) Find, in acres, the area of a triangle, if two of its sides are 16.342 rds. and 23.461 rds., and the included angle is 135°.

(4.) Find the area of the triangle in the preceding example in hectares.

(5.) The sides of a triangle are 13.461, 16.243, and 20.042 miles. Find the areas of the parts into which it is divided by any median.

(6.) The sides of a triangle are 12 in., 15 in., and 17 in. Find the areas of the parts into which it is divided by the bisector of the smallest angle.

(7.) Two sides of a triangle are in the ratio 2 to 5. Find the ratio of the parts into which the bisector of the included angle divides the triangle.

(8.) The altitude upon the hypotenuse of a right triangle is 98.423 in. One part into which the altitude divides the hypotenuse is four times the other. Find the area of the triangle.

(9.) Find the perimeter of the triangle in the preceding example.

(10.) The areas of two similar polygons are 22.462 sq. in. and 14.391 sq. m. A side of the first is 2 in. Find the homologous side of the second.

(11.) The sides of a triangle are .016256, .013961, and .020202. Find the radius of the inscribed circle.

(12.) A mirror measuring 33 in. by 22 in. is to have a frame of uniform width whose area is to equal the area of the mirror; find what the width of the frame should be.

(13.) The sum of the radii of the inscribed, circumscribed, and an escribed circle of an equilateral triangle is unity. What is the area of the triangle?

BOOK V

PROBLEMS OF DEMONSTRATION

120. An equilateral polygon inscribed in a circle is regular. An equilateral polygon circumscribed about a circle is regular, if the number of sides is odd.

121. An equiangular polygon inscribed in a circle is regular if the number of sides is odd. An equiangular polygon circumscribed about a circle is regular.

122. The diagonals of a regular pentagon are equal.

123. The pentagon formed by the diagonals of a regular pentagon is regular.

124. An inscribed regular octagon is equivalent to a rectangle whose sides are equal to the sides of an inscribed and a circumscribed square.

125. If a triangle is formed having as sides the radius of a circle, the side of an inscribed regular pentagon, and the side of an inscribed regular decagon, this triangle will be a right triangle.

126. The area of a regular hexagon inscribed in a circle is a mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

127. If perpendiculars are drawn from the vertices of a regular polygon to any straight line through its centre, the sum of those which fall upon one side of the line is equal to the sum of those which fall upon the other side.

128. The area of any regular polygon inscribed in a circle is a mean proportional between the areas of the inscribed and circumscribed polygons of half the number of sides.

129. If, on the sides of a right triangle as diameters, semi-circumferences are described exterior to the triangle, and a circumference is drawn through the three vertices, the sum of the crescents thus formed is equivalent to the triangle.

130. If two circles are internally tangent to a third circle and the sum of their radii is equal to the radius of the third circle, the shorter arc of the third circle comprised between their points of contact is equal to the sum of the arcs of the two small circles from their points of contact with the third circle to their intersection which is nearest the large circle.

131. If *CD* is the perpendicular from the vertex of the right angle of a right triangle *ABC*, prove that the areas of the circles inscribed in the triangles *ACD*, *BCD* are proportional to the areas of the triangles.

PROBLEMS OF CONSTRUCTION

132. To construct a circumference whose length shall equal the sum of the lengths of two given circumferences.

133. To construct a circle equivalent to the sum of two given circles.

134. To inscribe a regular octagon in a given square.

135. To inscribe a regular hexagon in a given equilateral triangle.

136. Divide a given circle into any number of parts proportional to given straight lines by circumferences concentric with it.

137. Find four circles whose radii are proportional to given lines, and the sum of whose areas is equal to the area of a given circle.

138. In a given equilateral triangle inscribe three equal circles each tangent to the two others and to two sides of the triangle.

139. In a given circle inscribe three equal circles each tangent to the two others and to the given circle.

140. The length of the circumference of a circle being represented by a given straight line, find approximately by a geometrical construction the radius.

PROBLEMS FOR COMPUTATION

141. (1.) A regular octagon is inscribed in a circle whose radius is 4 ft. Find the segment of the circle contained between one side of the octagon and its subtended arc.

(2.) Find the area of an equilateral triangle circumscribed about a circle whose radius is 14.361 in.

(3.) An isosceles right triangle is circumscribed about a circle whose radius is 3 cm. Find (*a*) each side; (*b*) its area; (*c*) the area in each corner of the triangle bounded by the circumference of the circle and two sides of the triangle.

(4.) Find the area of the circle inscribed in an equilateral triangle, one side of which is 7.4631 ft.

(5.) Find the difference between the area of a triangle whose sides are 4.6213 mm., 3.7962 mm., and 2.6435 mm., and the area of the circumscribed circle.

(6.) The area of a circle is 14632 sq. ft. Find its circumference in yards.

(7.) Find the area of a ring whose outer circumference is 15.437 ft., and whose inner circumference is 9.3421 ft.

(8.) Find the ratio of the areas of two circles inscribed in equilateral triangles, if the perimeter of one triangle is four times that of the other.

(9.) If the area of an equilateral triangle inscribed in a circle is 12 sq. ft., what is the area of a regular hexagon circumscribed about the same circle?

(10.) Find the side of a regular octagon whose area shall equal the sum of the areas of two regular hexagons, one inscribed in and the other circumscribed about a circle whose radius is 10.462 in.

(11.) A man has a circular farm 640 acres in extent. He gives to each of his four sons one of the four largest equal circular farms which can be cut off from the original farm. How much did each son receive?

(12.) A man has a circular tract of land 700 acres in area; he wills one of the three largest equal circular tracts to each of his three sons, the tract at the centre included between the three circular tracts to his daughter, and the tracts included between the circumference of the original tract and the three circular tracts to his wife. How much will each receive?

(13.) A man owned a tract of land 323,250 sq. m. in area, and in the form of an equilateral triangle. To each of his three sons he gave one of the three largest equal circular tracts which could be formed from the given tract; to each of his three daughters one of the corner sections cut off by a circular tract; to each of his three grandchildren one of the side sections cut off by two of the circular tracts; he himself retained the central section included between the three circular tracts. Find the share of each.

TABLE OF MEASURES AND WEIGHTS

English Measures

LENGTH

12 inches (in.)	= 1 foot (ft.).
3 feet	= 1 yard (yd.).
5½ yards	= 1 rod (rd.).
4 rods	= 1 chain (ch.).
80 chains	= 1 mile (m.).
1 yard	= .9144 meter.
1 mile	= 1.6093 kilometers.

SURFACE

144 sq. inches	= 1 sq. foot.
9 sq. feet	= 1 sq. yard.
30¼ sq. yards	= 1 sq. rod.
160 sq. rods	= 1 acre.
640 acres	= 1 sq. mile.
1 sq. yard	= 0.8361 sq. meter.
1 acre	= 0.4047 hectare.

VOLUME

1728 cu. inches	= 1 cu. foot.
27 cu. feet	= 1 cu. yard.
128 cu. feet	= 1 cord (cd.).
1 cu. yard	= 0.7646 cu. meter.
1 cord	= 3.625 steres.

ANGLES

60 seconds (″)	= 1 minute (′).
60 minutes	= 1 degree (°).
90 degrees	= 1 right angle.

CIRCLES

360 degrees = 1 circumference.

$\pi = 3.1416 =$ nearly $3^1/_7$.

CAPACITY

1 liq. gal. = 3.785	liters =	231 cu. in.
1 dry gal. = 4.404	liters =	268.8 cu. in.
1 bushel = 0.3524	hkl. =	2150.42 cu. in.

AVOIRDUPOIS WEIGHT

16 ounces (oz.)	= 1 pound (lb.).
100 lbs.	= 1 hundredweight (cwt.).
20 hundredweight	= 1 ton (T.).

1 pound = .4536 kilo. = 7000 grains.

1 ton = .9071 tonneau.

Metric Measures

LENGTH

10 millimeters (mm.)	= 1 centimeter (cm.).
10 centimeters	= 1 decimeter (dcm.).
10 decimeters	= 1 meter (m.).
10 meters	= 1 dekameter (dkm.).
10 dekameters	= 1 hektometer (hkm.).
10 hektometers	= 1 kilometer (km.).
1 meter	= 39.37 inches.
1 kilometer	= 0.6214 mile.

SURFACE

100 sq. millimeters	= 1 sq. centimeter.
100 sq. centimeters	= 1 sq. decimeter.
100 sq. decimeters	= {1 sq. meter. / 1 centare (ca.).
100 centares	= 1 are (a.).
100 ares	= 1 hektare (hka.).
1 sq. centimeter	= 0.1550 sq. inch.
1 sq. meter	= 1.196 sq. yards.
1 are	= 3.954 sq. rods.
1 hektare	= 2.471 acres.

VOLUME

1000 cu. millimeters	= 1 cu. centimeter.
1000 cu. centimeters	= 1 cu. decimeter.
1000 cu. decimeters	= 1 cu. meter.
	= 1 stere (st.).
1 cu. centimeter	= 0.061 cu. inch.
1 cu. meter	= 1.308 cu. yards.
1 stere	= 0.2759 cord.

CAPACITY

100 centiliters (cl.)	= 1 liter (l.).
100 liters	= 1 hektoliter (hkl.)

1 liter = 1.0567 liq. qts. = 1 cu. dcm.

METRIC WEIGHT

1000 grams (gm.)	= 1 kilogram (kilo.).
1000 kilograms	= 1 tonneau (t.).
1 gram	= 15.432 grains.
1 kilogram	= 2.2046 pounds.
1 tonneau	= 1.1023 tons.

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[The references are in general to sections.]

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